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Flexure and torsion locking phenomena in out-of-plane deformation of Timoshenko curved beam element

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ABSTRACT

The various locking phenomena in a linear Timoshenko curved beam element are identified in the context of out-of-plane deformations. A study on performance sensitivity of straight and curved Timoshenko beam finite elements to flexure-to-shear (*EI/GA*) and flexure-to-torsion (*EI/GJ*) stiffness ratios are carried out. The use of consistent field interpolation for shear strain is shown to eliminate shear locking effects, which depends on the ratio *EI/GA*, in both straight and curved beams. However, in the case of curved beam, a curvature related *additional spurious stiffening* which depends on *EI/GJ* ratio is observed. This additional stiffening effect is attributed to inconsistencies present in the out-of-plane flexure and torsion strain definitions, which are critically examined to characterize their adverse effects on the solution convergence. The results reveal the existence of two additional locking phenomena which we introduce here as *flexure locking* and *torsion locking*. The field consistency requirements to eliminate these locking effects have been identified. In the numerical examples, the field consistency in flexure and torsion strains is applied, independently and in combination, to understand the hidden perspectives of these locking phenomena. The regimes where these locking effects become significant are identified based on the relative magnitudes of flexural and torsional stiffnesses. The convergence characteristics in various locking regimes are studied for different models and their performances are discussed in detail.

1. Introduction

The design of general curved beams entails analysing a complex structural behaviour, unlike straight ones. The deformations of a curved beam are coupled due to active interactions among membrane, shear, flexure and torsion strain fields. In the displacement based finite element model, the use of low-order (especially, linear) polynomial interpolation for the displacement field variables participating in the strain definitions cause the element to 'lock' severely, affecting the convergence to the true solution.

In linear straight Timoshenko beam finite elements, only shear locking phenomenon is present. However, in curved Timoshenko beam finite elements, apart from shear locking, membrane locking also exists. Both shear and membrane locking in beam finite elements have been attributed to their inability to reproduce the 'shearless bending' and 'inextensible bending', respectively, in thin beams [1–3]. Several researchers have studied the locking associated with in-plane deformation of curved beams and proposed methods to eliminate the ill effects of shear and membrane locking [4–21]. These methods include reduced or selective integration [3,5], use of higher order interpolations [6–10], field consistency approach [11–14], formulations based on mixed/hybrid methods [15,16], penalty relaxation techniques [17], coupled displacement fields [18,19], and formulations based on assumed strains [20,21].

The literature on locking phenomena related to out-of-plane behaviour of curved beam elements is limited. Sabir [22] presented stiffness matrices for general deformation of Euler curved beam element using independent strain functions. Choi and Lim [23] developed linear and quadratic shear flexible curved beam elements. As the formulations are based on assumed strain fields they exhibited no locking effects. Prathap et al. [24] studied the consistency requirements of out-of-plane transverse shear strain field for a quadratic beam element based on Cartesian stain definitions. Using numerical studies they showed that full integration of this element does not suffer from locking. The literature review indicates that a detailed study on possible locking mechanisms in displacement based linear curved beam element has not been carried out for out-of-plane behaviour.

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In this article, we examine locking phenomena associated with out-of-plane deformation of a linear Timoshenko curved beam element. Presence of curvature related spurious mechanisms which depend on the flexure-to-torsion stiffness ratio are observed. The locking regimes based on the relative magnitudes of flexure and torsion stiffness are identified using appropriate numerical examples. Performance of different field consistent finite element models is also evaluated.

The remainder of this article is organized as follows: Section 2 introduces the geometry, coordinate systems and the basic equations associated with out-of-plane deformations for a linear Timoshenko curved beam element. Field consistency aspects are discussed in Section 3. Different field-consistent and field-inconsistent finite element models used in this study and their implementation are discussed in Section 4. In Section 5, numerical studies are presented to establish the presence of curvature-related locking phenomena. Based on the difference observed in the convergence characteristics of straight and curved beams of equal length, curvature related spurious stiffening effects are identified. Dependency of this additional stiffening on the flexure-to-torsion stiffness ratio is studied, based on which existence of two new locking phenomena is demonstrated for the first time. The effectiveness of consistent strain field interpolation in eliminating these locking effects and improving the convergence behaviour of the linear Timoshenko curved beam element is studied in detail. The distribution of stress resultants in the flexure and torsion locking regimes are presented for field consistent and field inconsistent models. Several conclusions are presented in Section 6.

2. Finite element formulation

The geometry of a linear curved beam element of length $2L$ and radius of curvature R is shown in Fig. 1. A right-handed orthogonal curvilinear coordinate system s–y–z is used with its origin placed at the centre of the element. The normalized coordinate ξ runs along the circumferential direction such that $\xi = s/L$.

The out-of-plane shear, flexure, and torsion strain components in the curvilinear coordinate system are:

$$\gamma = v' - \theta_z \tag{1}$$

$$\kappa = -\theta_z' + \frac{\theta_s}{R} \tag{2}$$

$$\tau = \theta_s' + \frac{\theta_z}{R} \tag{3}$$

In the above equations, v is the transverse displacement, θ_z is the flexure rotation, and θ_s is the torsion rotation of the cross-section. The prime ($'$) on a variable indicates its derivative with respect to the circumferential coordinate s.

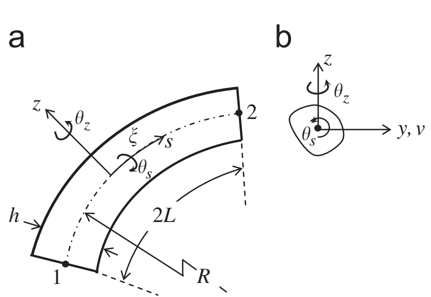

Fig. 1. Coordinate system for the linear curved beam element: (a) geometry and (b) arbitrary cross section.

The total potential energy functional Π_e of the beam element can be written as

$$\Pi_e(v,\theta_z,\theta_s) = \frac{1}{2}\int_{-L}^{L}\left[kGA(v'-\theta_z)^2 + EI\left(-\theta_z' + \frac{\theta_s}{R}\right)^2 + GJ\left(\theta_s' + \frac{\theta_z}{R}\right)^2\right]ds - \int_{-L}^{L} q(s)v\,ds - \int_{-L}^{L} m(s)\theta_z\,ds - \int_{-L}^{L} t(s)\theta_s\,ds - \sum_{i=1}^{6}\tilde{Q}_i^e \Delta_i^e \tag{4}$$

The terms in the square bracket represent the strain energy due to bending, transverse shear deformation, and torsion, respectively; q, m, and t are the distributed transverse load, bending moment, and torque along the span of the beam element; $\tilde{Q}_i^e$ and Δ_i^e are the generalized element forces and displacements; E and G denote Young's modulus and shear modulus, respectively, k_s is the shear correction factor; and R is the radius of curvature of the beam element. The geometric parameters A, I, and J denote the area, moment of inertia, and torsional constant for the cross-section, respectively.

The weak forms of the governing equations are obtained by applying principle of *minimum total potential energy*, $\delta\Pi_e = 0$ [25]:

$$0 = \int_{-L}^{L}\left[GAk_s\frac{d\chi}{ds}\left(\frac{dv}{ds} - \theta_z\right) - q(s)\chi\right]ds - (\tilde{Q}\chi)_{s=L} - (\tilde{Q}\chi)_{s=-L} \tag{5.1}$$

$$0 = \int_{-L}^{L}\left[-GAk_s\psi\left(\frac{dv}{ds} - \theta_z\right) + GJ\frac{\psi}{R}\left(\frac{d\theta_s}{ds} + \frac{\theta_z}{R}\right) - EI\frac{d\psi}{ds}\left(-\frac{d\theta_z}{ds} + \frac{\theta_s}{R}\right) - m(s)\psi\right]ds - (\tilde{M}\psi)_{s=L} - (\tilde{M}\psi)_{s=-L} \tag{5.2}$$

$$0 = \int_{-L}^{L}\left[GJ\frac{d\varphi}{ds}\left(\frac{d\theta_s}{ds} + \frac{\theta_z}{R}\right) - EI\frac{\varphi}{R}\left(-\frac{d\theta_z}{ds} + \frac{\theta_s}{R}\right) - t(s)\varphi\right]ds - (\tilde{T}\varphi)_{s=L} - (\tilde{T}\varphi)_{s=-L} \tag{5.3}$$

Eqs. (5.1)–(5.2) are the weak statements for Timoshenko curved beam element under out-of-plane deformation. $\tilde{Q}$, $\tilde{M}$, and $\tilde{T}$ denote internal generalized shear force, bending moment, and torque, respectively. The functions χ, ψ, φ denote the variations of out-of-plane transverse deflection v, flexure rotation θ_z, and torsion rotation θ_s, respectively.

Next, we perform integration by parts on Eqs. (5.1)–(5.3) to obtain the governing equations and associated natural boundary conditions:

$$0 = -\int_{-L}^{L}\left[GAk_s\chi\left(\frac{d^2v}{ds^2} - \frac{d\theta_z}{ds}\right)\right]ds + \left[GAk_s\chi\left(\frac{dv}{ds} - \frac{d\theta_z}{ds}\right)\right]_{-L}^{L} - \int_{-L}^{L} q(s)\chi\,ds - (\tilde{Q}\chi)_{s=L} - (\tilde{Q}\chi)_{s=-L} \tag{6.1}$$

$$0 = \int_{-L}^{L}\left[-GAk_s\psi\left(\frac{dv}{ds} - \theta_z\right) + GJ\frac{\psi}{R}\left(\frac{d\theta_s}{ds} + \frac{\theta_z}{R}\right) + EI\psi\left(-\frac{d^2\theta_z}{ds^2} + \frac{1}{R}\frac{d\theta_s}{ds}\right)\right]ds - \left[EI\psi\left(-\frac{d\theta_z}{ds} + \frac{\theta_s}{R}\right)\right]_{-L}^{L} - \int_{-L}^{L} m(s)\psi\,ds - (\tilde{M}\psi)_{s=L} - (\tilde{M}\psi)_{s=-L} \tag{6.2}$$

$$0 = \int_{-L}^{L}\left[-GJ\varphi\left(\frac{d^2\theta_s}{ds^2} + \frac{1}{R}\frac{d\theta_z}{ds}\right) - EI\frac{\varphi}{R}\left(-\frac{d\theta_z}{ds} + \frac{\theta_s}{R}\right)\right]ds + \left[GJ\varphi\left(\frac{d\theta_s}{ds} + \frac{\theta_z}{R}\right)\right]_{-L}^{L} - \int_{-L}^{L} t(s)\varphi\,ds - (\tilde{T}\varphi)_{s=L} - (\tilde{T}\varphi)_{s=-L} \tag{6.3}$$

Using Eqs. (6.1)–(6.3), the governing equations for out-of-plane deformation of a Timoshenko curved beam element in