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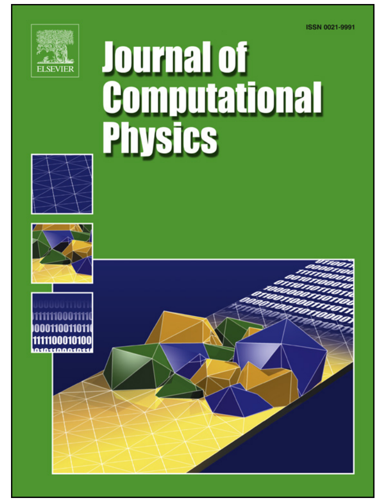

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Abstract

We describe an energy-enstrophy conserving discretisation for the rotating shallow water equations with slip boundary conditions. This relaxes the assumption of boundary-free domains (periodic solutions or the surface of a sphere, for example) in the energy-enstrophy conserving formulation of McRae and Cotter (2014). This discretisation requires extra prognostic vorticity variables on the boundary in addition to the prognostic velocity and layer depth variables. The energy-enstrophy conservation properties hold for any appropriate set of compatible finite element spaces defined on arbitrary meshes with arbitrary boundaries. We demonstrate the conservation properties of the scheme with numerical solutions on a rotating hemisphere.

1 Introduction

For large scale balanced flows, energy and enstrophy are important quantities for the rotating shallow water equations due to the cascade of energy to large scales whilst enstrophy cascades to small scales. At the level of numerical discretisations, energy conservation becomes important over long time integrations, whilst enstrophy conservation (or dissipation at the small scale) provides control of the regularity of the velocity field over long times.

Energy and enstrophy conserving schemes for the rotating shallow water equations have a long history that goes back to Arakawa and Lamb (1981); Sadourny (1975). The finite difference schemes in these papers were constructed from two important ingredients: (1) the vector-invariant form of the equations, and (2) the use of staggered grid finite difference methods built around discretisations of the div, grad and curl operators that preserve the vanishing div-curl and curl-grad identities at the discrete level. These discretisations form the foundations of several operational weather, ocean and climate models that are in current use. Another important practical aspect is that discretisations should preserve stationary geostrophic modes when applied to the f-plane linearisation of the shallow water equations. Ringler et al. (2010) addressed the issue of extending these properties to C-grid staggered finite difference discretisations on unstructured orthogonal grids, describing separate energy-conserving and enstrophy-conserving schemes; Thuburn and Cotter (2012) extended these ideas to non-orthogonal grids, making use of ideas from discrete exterior calculus (Hirani, 2003). Ringler et al. (2010) also considered enstrophy dissipation through the Anticipated Potential Vorticity method, following the structured rectangular grid formulation of Arakawa and Hsu (1990). There is still no known closed form for an energy-enstrophy conserving C-grid formulation on unstructured grids with an f-plane linearisation that preserves stationary geostrophic modes, but Eldred and Randall (2017) showed that such schemes can be obtained computationally through numerical optimisation.

In a series of papers, Salmon (2004, 2005, 2007), Salmon showed how to use Poisson and Nambu brackets to build conservation into numerical discretisations. For example, Stewart and Dellar (2016) provided a C-grid discretisation for the multi-layer shallow-water equations with complete Coriolis force. The variational formulation finite element method makes it easier to mimic the Poisson bracket structure of the vector-invariant shallow water equations at the discrete level, whilst compatible finite element spaces replicate the div-curl and curl-grad identities in the discrete setting. McRae and Cotter (2014) showed that this leads to a natural energy-enstrophy conserving compatible finite element scheme, with the bracket structure being exposed in the appendix. The finite element exterior calculus framework underpinning these properties was exposed by Cotter and Thuburn (2014). The same structure has been exploited to produce energy-enstrophy conserving discretisations using more exotic finite element spaces. Eldred et al. (2016) constructed compatible spaces from splines that allow higher-order