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A fourth-order scheme for space fractional diffusion equations

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ABSTRACT

A weighted and shifted difference formula is constructed based on the Lubich operators, which gives a forth-order and unconditionally stable difference scheme for the Cauchy problem of space fractional diffusion equations. The novelty of the proposed method here is that only four weighted parameters are required, compared to eight parameters used in the previous work, to achieve the fourth-order accuracy and to ensure the stability at the same time. To verify the efficiency of the proposed scheme, several numerical experiments for both one-dimensional and two-dimensional fractional diffusion problems are provided.
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1. Introduction

Fractional derivatives turn out to be important tools nowadays since they find applications and are superior to the traditional integer derivatives in modeling many problems, which include contaminant flow in heterogenous porous media [1,2], acoustical wave in complex media [13,21], fractional quantum mechanics [15], viscoelastic damping [18], etc. Due to the non-local feature of fractional operators, it is challenging and sometimes even impossible to evaluate the fractional derivatives for most functions, and the exact solutions of fractional diffusion equations are hardly to be given either. Hence efficiently solving fractional diffusion equations by numerical methods is drawing more and more attentions in research and practice.

From the perspective of the numerical analysis, there are some substantial difficulties in approximating the fractional derivatives. The finite difference method is one of the most commonly used methods in solving the fractional diffusion problems over the last decades, see [5,17,20]. Since the Riemann–Liouville (R–L) derivative and Grünwald–Letnikov (G–L) derivative are theoretically equivalent under the smooth assumptions imposed on the initial value [22], the G–L definition is commonly used to approximate the R–L derivative, and this approximation has a first-order accuracy. Nevertheless, when applying the approximation to the space fractional differential equations, it leads to an unstable scheme. To address such a problem, a shifted Grünwald formula was proposed by Meerschaert and Tadjeran [17] to approximate the space fractional derivatives, and successfully applied to solve a class of advection–dispersion equations [19,20,24,25]. Again, this shifted difference formula has only first-order accuracy in space. Cuesta, Lubich, and Palencia discussed the convolution quadrature time discretization of fractional diffusion-wave equations in [8], however, when applying the shifted Lubich's formula to ensure the stability of the numerical scheme, it also reduces to the first-order accuracy.

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In order to improve the efficiency of computation and to solve high-dimensional problems, many researchers devoted to developing high-order finite difference methods for solving the space fractional diffusion equations. Recently, Deng's group constructed a class of second-order finite difference schemes by using the weighted and shifted Grünwald difference (WSGD) for the approximations [26]. Moreover, adopting the same idea and with a compact technique, they also obtained a third-order scheme, namely, the quasi-compact scheme [28]. Based on Lubich's high-order operators [16], Deng's group further derived a class of second-, third-, and fourth-order difference approximations for the R–L fractional derivatives [6,7]. However, to obtain a fourth-order accuracy, they need to use eight distinct difference formulas for the weighted average. Hao et al. [12] proposed another method to get a fourth-order finite difference scheme with only three weight parameters, while this method requires demanding boundary conditions and is hardly applied to a high-order (higher than fourth-order) scheme. It worth mention that Sun's group have also worked on the higher-order finite difference schemes by Lubich's difference operators for time dependent fractional problems [10,14].

In this paper, we focus on developing a new fourth-order finite difference approximation for solving the Cauchy problem of space fractional diffusion equations. Compared to the method using eight weight parameters for a fourth-order scheme in [6], the novelty of the present work is that we use only four weight parameters to achieve the same accuracy. In addition, it can be straightforwardly extended to even higher-order finite difference formulas by adopting the same idea. The numerical schemes proposed are shown to be unconditionally stable with fourth-order accuracy in space for the Cauchy problems of both 1-D and 2-D fractional diffusion equations. It is worth to mention that, the solution of the fractional diffusion equation usually has singularities at the boundaries when solving an initial boundary value problem. While our difference scheme here requires regularities of the solutions, and hence cannot achieve a fourth-order convergence rate in such a case. The investigation of this problem will be studied in a future work.

The rest part of the present paper is organized as follows. In Section 2, we propose a class of finite difference operators to approximate the R–L fractional derivatives with fourth-order truncation error. In Section 3, one-dimensional and two-dimensional fractional diffusion equations are solved numerically by using Crank–Nicolson scheme and the alternating directional implicit (ADI) method for time discretization respectively. Stability analyses for both 1-D and 2-D cases are also discussed. In order to verify the accuracy and efficiency of the present method, we also provide several numerical experiments in Section 4. Finally, the concluding remarks are given in the last Section.

2. Fourth-order discretizations for space fractional operators

In this section, we develop a class of fourth-order approximations for the Riemann–Liouville fractional derivatives and show that they are efficient in solving space fractional diffusion PDEs, *i.e.*, all the eigenvalues of the matrices corresponding to the discretized operators have negative real parts.

2.1. Derivation of the discretization scheme

We start with the definitions of the Riemann–Liouville derivatives and their Fourier transforms.

Definition 1. The left- and right-sided Riemann–Liouville fractional derivatives of order α are defended by

$$ {}_{-\infty}D_x^\alpha u(x) := \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{-\infty}^{x} \frac{u(y)}{(x-y)^{\alpha+1-n}} \mathrm{d}y, \tag{1} $$

$$ {}_{x}D_\infty^\alpha u(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_{x}^{\infty} \frac{u(y)}{(y-x)^{\alpha+1-n}} \mathrm{d}y, \tag{2} $$

respectively, where $n = \lfloor \alpha \rfloor + 1$ is the smallest integer exceeding α.

Lemma 2. *Let $1 < \alpha \le 2$, the Fourier transforms of the left- and right-sided Riemann–Liouville derivatives satisfy*

$$ \mathcal{F}\left({}_{-\infty}D_x^\alpha u(x)\right) = (i\xi)^\alpha \hat{u}(\xi), \qquad \mathcal{F}\left({}_{x}D_\infty^\alpha u(x)\right) = (-i\xi)^\alpha \hat{u}(\xi), \tag{3} $$

where $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$.

Using the fractional linear multi-step methods, Lubich [16] obtained L-th order ($L \le 6$) approximations of the α-th derivative ($\alpha > 0$) or integral ($\alpha < 0$) based on the coefficients of the corresponding generating function $\delta^\alpha(z)$, where

$$ \delta^\alpha(z) = \left(\sum_{i=1}^{L} \frac{1}{i}(1-z)^i \right)^\alpha . \tag{4} $$

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