Journal of Computational Physics 373 (2018) 648–661

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Direct sampling method for retrieving small perfectly conducting cracks

Won-Kwang Park

Department of Information Security, Cryptology, and Mathematics, Kookmin University, Seoul, 02707, Republic of Korea

ARTICLE INFO

Article history:
Received 20 February 2018
Received in revised form 5 June 2018
Accepted 7 July 2018
Available online xxxx

Keywords:
Direct sampling method
Perfectly conducting crack
Bessel functions
Numerical experiments

ABSTRACT

In this paper, direct sampling method is considered for determining the location of a set of small, linear perfectly conducting cracks from the collected far-field data corresponding to an incident field. To show the feasibility of the direct sampling method, this study proves that the indicator function of the direct sampling method can be represented by the Bessel function of order zero and the crack lengths. The results of the numerical simulations are shown to support the fact that the imaging performance is highly dependent on the crack lengths. To explain the fact that the imaging performance is highly dependent on the rotation of the cracks, the direct sampling method is further analyzed by establishing a representation using Bessel functions of orders zero and one. Based on the derived representation of indicator function, we design improved direct sampling methods by applying incident fields with multiple directions and multiple frequencies. Corresponding analysis of indicator functions and simulation results are shown for demonstrating the effectiveness and improvements.

1. Introduction

This study considers a direct sampling method for fast imaging of small and linear perfectly conducting cracks located in a two-dimensional space $\mathbb{R}^2$. It is well known that the direct sampling method is a fast, simple, and effective imaging technique. Furthermore, it requires only a few (one or two) incident fields and does not require additional operations, e.g., singular value decomposition, adjoint problem solving, or ill-posed integral equations. Therefore, the direct sampling method is applied to many inverse scattering problems in two- and three-dimensional problems [1–3], source detection in stratified ocean waveguides [4], electrical impedance tomography [5], diffusive optical tomography [6], and microwave imaging [7].

Based on these studies, the direct sampling method can be demonstrated to be effective for full-view inverse scattering problems. In particular, based on the relation between the Bessel function of order zero and the indicator function of the direct sampling method [1,3], the reason for the detection of targets was investigated. However, the analysis was not fully reliable in the imaging of the cracks. For example, cracks that were significantly shorter than others were theoretically undetectable and their identified locations were different corresponding to the direction of propagation. Hence, a further analysis of the indicator function for the direct sampling method is still required to be performed, which is the motivation for this research.

This study identifies the mathematical structure of the indicator function of the direct sampling method. The study proves that the indicator function can be represented by the Bessel functions of order zero and one, the length and rotation

E-mail address: parkwk@kookmin.ac.kr.

https://doi.org/10.1016/j.jcp.2018.07.014

of the cracks, and the direction of propagation of incident field. This is based on the fact that the far-field pattern can be represented by an asymptotic expansion formula in the presence of small and linear perfectly conducting cracks [8]. From the identified structure, this study explains the reason for the unexplained phenomenon and finds two improvement methods by employing multiple incident fields and frequencies. In the analysis and numerical experiments, we demonstrated an improvement in the direct sampling method theoretically and numerically.

This remainder of this study is organized as follows. In Section 2, this study surveys the two-dimensional forward problem, asymptotic expansion formula, and indicator function for the direct sampling method. In Section 3, this research identifies the structure of the indicator function by establishing a relation with the Bessel functions of orders zero and one, the length and rotation of the cracks, and the incident field direction. The identification of the direct sampling method is highly dependent on not only the length and rotation of the cracks but also on the direction of the incident field. Several results of numerical simulations were found to support identified structure of the indicator function. In Section 4, this study introduces two improvement methods by employing multiple directions of the incident fields and multiple frequencies. Furthermore, numerical simulations were performed to examine the improvement. Section 5 contains a short conclusion and some proposals for future work.

2. Forward problem and direct sampling method

2.1. Two-dimensional forward problem and far-field pattern

In this section, the two-dimensional direct scattering problem is introduced for M different well-separated linear perfectly conducting cracks of length $2\ell_m$, which are denoted by Σ_m, $m = 1, 2, \cdots, M$ and located in the homogeneous space $\mathbb{R}^2$. A more detailed description can be found in a previous study [9]. Throughout this study, Σ_m is denoted as follows

$$\Sigma_m = \left\{\mathcal{R}_\phi[x_m + h, y_m]^{\mathrm{T}} : -\ell_m \le h \le \ell_m\right\},$$

for $m = 1, 2, \cdots, M$ and Σ is assumed to be the collection of cracks. Here, $\mathcal{R}_\phi$ denotes the rotation by ϕ and we denote $\mathbf{c}_m = \mathcal{R}_\phi[x_m, y_m]^{\mathrm{T}}$ be the center of Σ_m. This study assumes that Σ_m are sufficiently separated from each other such that

$$k|\mathbf{c}_m - \mathbf{c}_{m'}| \gg 1 - \frac{1}{4} = \frac{3}{4},$$

where k denotes the positive wavenumber, which is of the form $k = 2\pi/\lambda$. Here, λ is the given wavelength and it is assumed that $2\ell_m \ll \lambda$ for all $m = 1, 2, \cdots, M$. Based on a previous study [3], this research considers the plane-wave illumination $\psi_{\mathrm{inc}}(\mathbf{x}, \mathbf{d}) = e^{ik\mathbf{d}\cdot\mathbf{x}}$ as the given incident field with a fixed propagation direction $\mathbf{d} \in \mathbb{S}^1$. Here, $\mathbb{S}^1$ denotes a two-dimensional unit circle centered at the origin. Let $\psi(\mathbf{x}, \mathbf{d})$ be the time-harmonic total field. Then, based on mathematical treatment of the scattering of time-harmonic electromagnetic waves from thin infinitely long cylindrical obstacles, $\psi(\mathbf{x}, \mathbf{d})$ satisfies the Helmholtz equation

$$\triangle\psi(\mathbf{x}, \mathbf{d}) + k^2\psi(\mathbf{x}, \mathbf{d}) = 0 \quad \text{in} \quad \mathbb{R}^2\backslash\overline{\Sigma}, \tag{1}$$

with the Dirichlet boundary condition

$$\psi(\mathbf{x}, \mathbf{d}) = 0 \quad \text{on} \quad \Sigma. \tag{2}$$

Notice that the boundary condition (2) corresponds to scattering from a sound-soft arc, whereas in electromagnetics it is known as the Transverse Magnetic (TM) polarization, refer to [9].

Let $\psi_{\mathrm{scat}}(\mathbf{x}, \mathbf{d}) = \psi(\mathbf{x}, \mathbf{d}) - \psi_{\mathrm{inc}}(\mathbf{x}, \mathbf{d})$ be the scattered field $\psi_{\mathrm{scat}}(\mathbf{x}, \mathbf{d})$ that uniformly satisfies the Sommerfeld radiation condition

$$\lim_{|\mathbf{x}|\to\infty}\sqrt{|\mathbf{x}|}\left(\frac{\partial\psi_{\mathrm{scat}}(\mathbf{x}, \mathbf{d})}{\partial|\mathbf{x}|} - ik\psi_{\mathrm{scat}}(\mathbf{x}, \mathbf{d})\right) = 0$$

in all the directions $\boldsymbol{\theta} = \mathbf{x}/|\mathbf{x}|$. $\psi_\infty(\boldsymbol{\theta}, \mathbf{d})$ is denoted as the far-field pattern of $\psi_{\mathrm{scat}}(\mathbf{x}, \mathbf{d})$ that uniformly satisfies

$$\psi_{\mathrm{scat}}(\mathbf{x}, \mathbf{d}) = \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}}\left\{\psi_\infty(\boldsymbol{\theta}, \mathbf{d}) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right)\right\}, \quad |\mathbf{x}| \longrightarrow +\infty$$

in all the directions $\boldsymbol{\theta} = \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^1$. Based on a previous study [9], $\psi_\infty(\boldsymbol{\theta}, \mathbf{d})$ can be represented as the following single-layer potential with the unknown density function $\varphi(\mathbf{c}_m, \mathbf{d})$:

$$\psi_\infty(\boldsymbol{\theta}, \mathbf{d}) = -\frac{1+i}{4\sqrt{\pi k}}\sum_{m=1}^{M}\int_{\Sigma_m} e^{-ik\boldsymbol{\theta}\cdot\mathbf{c}_m}\varphi(\mathbf{c}_m, \mathbf{d})d\mathbf{c}_m. \tag{3}$$

Based on another study [8], the far-field pattern $\psi_\infty(\boldsymbol{\theta}, \mathbf{d})$ can be represented as the following asymptotic expansion formula, which plays a key role in the analysis of the imaging function of the direct sampling method.