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Anisotropic radial basis function methods for continental size ice sheet simulations

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ABSTRACT

In this paper we develop and implement anisotropic radial basis function methods for simulating the dynamics of ice sheets and glaciers. We test the methods on two problems: the well-known benchmark ISMIP-HOM B that corresponds to a glacier size ice and a synthetic ice sheet whose geometry is inspired by the EISMINT benchmark that corresponds to a continental size ice sheet. We illustrate the advantages of the radial basis function methods over a standard finite element method. We also show how the use of anisotropic radial basis functions allows for accurate simulation of the velocities on a large ice sheet, which was not possible with standard isotropic radial basis function methods due to a small ratio between the typical thickness and length of an ice sheet. Additionally, we implement a partition of unity method in order to improve the computational efficiency of the radial basis function methods.

1. Introduction

There is a growing interest in simulating the evolution of ice sheets for predicting their contribution to the future sea level rise [1] and also understanding the processes of forming past landscapes [2]. Mathematical models are introduced as tools to understand the dynamics of ice sheets in the past and in the future [3]. Improvement in accuracy and efficiency of the modeling and numerical methods is always needed, especially for large scale and long time simulations [4–6].

The ice flow is generally described as an incompressible, non-Newtonian fluid with highly nonlinear viscosity. The most commonly used model for simulating the ice dynamics is based on the so-called Stokes equations [7–9]. The deformation of the ice body under its own weight is governed by Glen's flow law [10]. It relates the stress field to the strain rates as a viscous fluid and the viscosity depends nonlinearly on the velocities, which introduces an additional degree of difficulty in the numerical solving procedure for the Stokes equations. Moreover, the Stokes system is a non-linear saddle-point problem, which requires special numerical treatments in a setting for finite element methods to satisfy the inf-sup condition [11], such as adding stabilization on the pressure variables [12] or using high-order finite element methods [13].

More precisely, the Stokes equations give rise to a three dimensional saddle point problem. Therefore, several simplifications are derived for the Stokes equations to reduce the computational complexity. The first order Stokes model (also known as the Blatter–Pattyn model) is based on the assumption that the hydrostatic pressure is balanced by the vertical normal stress [14,15], such that the horizontal gradient of the vertical velocity is neglected. The system is simplified to a three

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dimensional elliptic problem that only contains the horizontal velocities as unknowns and the vertical velocity is recovered by solving the incompressibility equation. Other approximations are the Shallow Ice Approximation (SIA) [16] and Shallow Shelf Approximation (SSA) [17,18], which simplify the three-dimensional problem to one-dimensional and two-dimensional, respectively. However, these approaches give lower order of approximations than the first order Stokes model, i.e., have a larger model error.

These models have been intercompared within several benchmark experiments during the past decade. For instance, some of the well-known benchmark experiments are the Ice Sheet Model Intercomparison Project for Higher-Order ice sheet Models (ISMIP-HOM) [5] and the framework of European Ice Sheet Modeling Initiative (EISMINT) [19]. In ISMIP-HOM, different Stokes approximations are compared on glacier size problems (about 10 km long), whereas in EISMINT, the computational domains are continental size (more than 1000 km long).

Traditional numerical methods such as finite element methods (FEM) are commonly used for solving ice sheet models since FEM can easily handle complex geometry with different types of boundary conditions. However, it has some drawbacks when solving problems on a domain with a moving boundary, which leads to remeshing of the entire domain for every time step. Also, solving the nonlinear system requires a full matrix reassembly during each nonlinear iteration. Therefore, a mesh-free method that can be stated in strong form is considered to be a preferred choice for such problems. Radial basis function (RBF) methods are of that kind [20,21]. The idea is to define a finite-dimensional basis, which consists of functions, whose values depend on the distance from their centers, and use them for approximating the solution.

RBF methods were first introduced for solving partial differential equations in the works of Kansa [22,23], where he used a global RBF method with basis functions globally supported over the entire computational domain. However, this approach results in a dense and often ill-conditioned coefficient matrix of the system of equations that needs to be solved. This makes the use of iterative solvers impossible, which prevents the method from an extension to higher dimensions. To overcome this issue, we use a localized RBF method, namely the radial basis function partition of unity method (RBF-PUM) [24–26], which is discussed in detail in Section 3.3. Another popular localized method is the radial basis function generated finite difference method (RBF-FD) [27–30], which can be seen as a generalization of finite difference methods. Recent developments of RBF-FD [31,32] showed that polyharmonic splines augmented with polynomials can be a powerful alternative to infinitely smooth RBFs. In this setting the polynomial degree, instead of the RBF, dictates the rate of convergence. A key advantage of this approach is that high accuracy typically does not lead to ill-conditioning. It is however still unclear if this approach combines well with the partition of unity method.

For ice simulations RBF methods have been introduced in [33], where the authors have studied advantages of an RBF approach for the Haut Glacier d'Arolla, which is also a test case from the ISMIP-HOM benchmark. In this paper, we continue the work and extend the approach to solve for dynamics of continental size ice sheets. We introduce anisotropic RBF approximations to solve problems with continental ice sheet geometries, which have small ratios between typical thickness and length. A small aspect ratio is an obstacle for standard isotropic RBFs due to the strong dependence of the shape parameter, which determines the width of the functions. The use of anisotropic RBFs significantly relaxes this dependence and simplifies the method implementation. Additionally, we suggest a strategy of selecting the shape parameter value based on the conditioning of the interpolation matrix. In order to enable more efficient simulations, especially towards three-dimensional problems, we develop a partition of unity approach that is also based on anisotropic patches. Also, we study error estimates for the anisotropic method and perform convergence tests.

The remainder of the paper is structured as follows. In Section 2 we explain the Stokes equations and the first order Stokes model that governs the dynamics of ice sheets. Then, in Section 3 we introduce an anisotropic RBF methodand we present numerical results in Section 4. We also provide a comparison of the RBF method with a standard FEM in terms of time-to-accuracy, i.e., we compare the execution times to achieve certain error levels. Finally, we draw some conclusions in Section 5.

2. Ice sheet dynamics

Ice can be viewed as a very viscous fluid flow on a large scale, for example, see Fig. 1. Thus, the models for simulating ice sheet dynamics are based on fluid dynamics laws. In general, ice is considered as an incompressible flow with a low Reynolds number and with the stress tensor related to the strain rate by a power law viscous rheology [7]. Due to the slow motion of ice masses, the acceleration term can be neglected and the Navier–Stokes equations can be turned into the Stokes equations that describe a steady flow. However, the computational demand for solving the Stokes system can be too high, especially for such large domains as ice sheets. Therefore, under assumptions of a small ratio between the ice thickness and length, the horizontal variations of the vertical velocity in the Stokes equations can be neglected. Thereby, we arrive at the simplified first order Stokes equations, which are an approximation to the Stokes equations in terms of the thickness/length ratio. The solution obtained by the first order Stokes equations is accurate in most of the area on the ice, but it may not be applicable at some transition zones where the slip boundary condition changes from no slip to free slip within a distance at the scale of the ice thickness, since the hydrostatic assumption of the vertical normal stress is not valid in this area [34,35].