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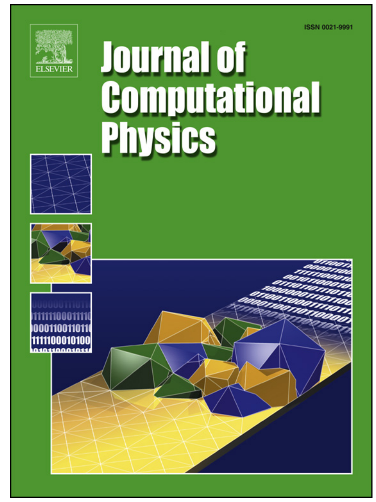

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A spectral method for nonlocal diffusion operators on the sphere

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Abstract

We present algorithms for solving spatially nonlocal diffusion models on the unit sphere with spectral accuracy in space. Our algorithms are based on the diagonalizability of nonlocal diffusion operators in the basis of spherical harmonics, the computation of their eigenvalues to high relative accuracy using quadrature and asymptotic formulas, and a fast spherical harmonic transform. These techniques also lead to an efficient implementation of high-order exponential integrators for time-dependent models. We apply our method to the nonlocal Poisson, Allen–Cahn and Brusselator equations.

Keywords:
nonlocal PDEs on the sphere, nonlocal diffusion operators, spectral methods, fast spherical harmonics, exponential integrators, pattern formation
2010 MSC: 33C55, 42B37, 65D30, 65L05, 65M20, 65M70

1. Introduction

Nonlocal models have been extensively studied in many fields such as materials science, thermodynamics, fluid dynamics, fracture mechanics, biology and image analysis [1, 2, 3, 4, 5, 6]. Many of these models can be conveniently formulated using nonlocal integral operators generalizing the standard differential operators of vector calculus [7, 8]. In this paper, we propose a fast spectral method for computing solutions of nonlocal models of the form

$$u_t = \epsilon^2 \mathcal{L}_\delta u + \mathcal{N}(u), \quad u(t=0, \mathbf{x}) = u_0(\mathbf{x}), \quad \epsilon > 0, \tag{1}$$

where $u(t, \mathbf{x})$ is a function of time $t \geq 0$ and position $\mathbf{x}$ on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, $\mathcal{N}$ is a nonlinear operator with constant coefficients (e.g., $\mathcal{N}(u) = u - u^3$), and $\mathcal{L}_\delta$ is a nonlocal *Laplace–Beltrami operator*,

$$\mathcal{L}_\delta u(\mathbf{x}) = \int_{\mathbb{S}^2} \rho_\delta(|\mathbf{x} - \mathbf{y}|) \left[u(\mathbf{y}) - u(\mathbf{x})\right] \, \mathrm{d}\Omega(\mathbf{y}). \tag{2}$$

In the definition (2) above, $|\mathbf{x} - \mathbf{y}|$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, $\mathrm{d}\Omega(\mathbf{y})$ denotes the standard measure on $\mathbb{S}^2$ and ρ_δ is a suitably defined nonlocal kernel with horizon $0 < \delta \leq 2$, which determines the range of interactions. $\mathcal{L}_\delta$ is also called a nonlocal diffusion operator on the sphere. The function u can be real or complex and the equation (1) can be a single equation as well as a system of equations.

There has been substantial work on the numerical approximation of the equation (1) in Euclidean domains [9, 10], including a spectral method for nonlocal diffusion operators defined over a periodic cell in $\mathbb{R}^d$ ($d \leq 3$) [3, 11]. However, no study has been attempted so far to investigate similar discretizations on the sphere. It is of practical interests to study the extension to non-Euclidean geometries with the sphere being a representative example, e.g., for the modelling of anomalous diffusion, pattern formation and image analysis.

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