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Robust boundary conditions for stochastic incompletely parabolic systems of equations

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ABSTRACT

We study an incompletely parabolic system in three space dimensions with stochastic boundary and initial data. We show how the variance of the solution can be manipulated by the boundary conditions, while keeping the mean value of the solution unaffected. Estimates of the variance of the solution is presented both analytically and numerically. We exemplify the technique by applying it to an incompletely parabolic model problem, as well as the one-dimensional compressible Navier–Stokes equations.

1. Introduction

The study of stochastic partial differential equations, with uncertainty in the boundary and initial data, is an important task in climatology [1], turbulent combustion [2], flow in porous media [3,4], electromagnetics [5,6] and seismology [7,8] to name a few examples.

There are essentially two types of techniques that are used to quantify the uncertainty in the solution. Intrusive methods [9–14] are based on polynomial chaos expansions, where the solution is expressed as a spectral expansion of the random variables. The method is cheap in terms of function evaluations for smooth problems, but requires a code, dealing specifically with the uncertainty. Non-intrusive methods are sample-based, i.e. the problem is solved several times for given values of the stochastic variables [15–18]. Quadrature techniques, often combined with sparse grid techniques [19,20], are used to determine the statistical moments of the solution. The same code that deals with the deterministic case can be used.

The focus of this paper will be on how the choice of boundary conditions contributes to the size and distribution of the uncertainty in the solution. In [21], these effects were analyzed for hyperbolic problems and applied to the Euler and Maxwell's equations. In this paper we extend the analysis to include incompletely parabolic problems, such as the compressible Navier–Stokes equations. The analysis is valid both for intrusive and non-intrusive techniques.

In the first part of the paper, we analyze how the boundary conditions affect the stochastic properties of the solution by studying the energy rate of the solution. It is shown that the energy rate is equivalent to an evolution equation for the variance [21]. We proceed and study the possibility of reducing the uncertainty in the solution by choice of boundary conditions.

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In the second part of the paper we make sure that the mathematical and numerical treatments of our initial boundary value problem is correct. Well-posed boundary conditions [22–26], are derived. To discretize in space we use a finite difference scheme with summation-by-parts (SBP) operators [27–29] and impose boundary conditions weakly by the Simultaneous-Approximation-Term (SAT) technique [30–33]. We show that a convergent scheme is obtained. Quadrature rules based on the probability distributions are used to non-intrusively compute the statistical moments of the solution [34–37].

In the third part of the paper, we exemplify the theoretical developments by numerical calculations. Numerical experiments are provided for a model problem and as a more realistic application, the compressible Navier–Stokes equations.

The rest of the paper will proceed as follows. In Section 2, the stochastic properties of the problem are analyzed, and estimates of the variance of the solution are derived. A model problem is studied in Section 4 where the implications of the technique are analyzed and discussed. In Section 5, the technique is applied to the one-dimensional Navier–Stokes equations. Finally, conclusions are drawn in Section 6.

2. The stochastic formulation

Consider the following system of equations,

$$\begin{aligned} u_t + Au_x + Bu_y + Cu_z &= F_x(u) + G_y(u) + H_z(u), && \vec{x} \in \Omega, \quad t \geq 0, \\ Lu &= g(\vec{x}, t, \vec{\xi}), && \vec{x} \in \partial\Omega, \ t \geq 0, \\ u &= f(\vec{x}, \vec{\xi}), && \vec{x} \in \Omega, \quad t = 0, \end{aligned} \tag{1}$$

where,

$$\begin{aligned} F(u) &= D_{11}u_x + D_{12}u_y + D_{13}u_z, \\ G(u) &= D_{21}u_x + D_{22}u_y + D_{23}u_z, \\ H(u) &= D_{31}u_x + D_{32}u_y + D_{33}u_z. \end{aligned} \tag{2}$$

The solution is represented by the vector $u = u(\vec{x}, t, \vec{\xi}) = [u_1, \ldots, u_M]$, where, $\vec{x} = (x, y, z)$, and $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_{M_\xi})$ is the vector of variables describing the stochastic variation of the problem. The $M \times M$ matrices A, B, C, D_{ij} are constant and symmetric. The block matrix with entries D_{ij} is assumed to be positive semi-definite, leading to an incompletely parabolic problem. L is the boundary operator defined on the boundary $\partial\Omega$. $f(\vec{x}, \vec{\xi})$ and $g(\vec{x}, t, \vec{\xi})$ are the stochastic initial and boundary data to the problem. With a proper choice of the matrices involved, the problem (1) represents the linearized and symmetrized compressible Navier–Stokes equations [38].

In the following form of (1), we highlight the stochastic nature of the data,

$$\begin{aligned} u_t + Au_x + Bu_y + Cu_z &= F_x(u) + G_y(u) + H_z(u), \\ Lu &= g(\vec{x}, t, \vec{\xi}) = \mathbb{E}[g] + \delta g(\vec{x}, t, \vec{\xi}), \\ u &= f(\vec{x}, \vec{\xi}) \quad\ = \mathbb{E}[f] + \delta f(\vec{x}, \vec{\xi}), \end{aligned} \tag{3}$$

where we for ease of notation have excluded the spatial and temporal domains where the equations are defined. The expected value is denoted by $\mathbb{E}[\psi(\cdot, \xi)] = \int_{\Omega_\xi} \psi(\cdot, \xi)\rho(\xi)\, d\xi$, where Ω_ξ is the stochastic domain and $\rho(\xi)$ the probability density function of ξ. Further, δ denotes the deviation from the mean. By taking the expected value of (3) and defining $v = \mathbb{E}[u]$ we obtain,

$$\begin{aligned} v_t + Av_x + Bv_y + Cv_z &= F_x(v) + G_y(v) + H_z(v), \\ Lv &= \mathbb{E}[g], \\ v &= \mathbb{E}[f], \end{aligned} \tag{4}$$

where we have used that F_x, G_y and H_z are linear in u.

Remark 1. Equation (4) is equivalent to the equation determining the first term in a Polynomial Chaos expansion of u, see [39].

Next, the difference between (3) and (4) together with the notation $e = u - v$ gives,

$$\begin{aligned} e_t + Ae_x + Be_y + Ce_z &= F_x(e) + G_y(e) + H_z(e), \\ Le &= \delta g(\vec{x}, t, \vec{\xi}), \\ e &= \delta f(\vec{x}, \vec{\xi}). \end{aligned} \tag{5}$$