Journal of Computational Physics 371 (2018) 280–314

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Conservative model reduction for finite-volume models

Kevin Carlberg [*,1], Youngsoo Choi [2], Syuzanna Sargsyan [3]

Sandia National Laboratories, United States of America

ARTICLE INFO

Article history:
Received 1 December 2017
Received in revised form 9 May 2018
Accepted 10 May 2018
Available online xxxx

Keywords:
Nonlinear model reduction
Structure preservation
Finite-volume method
Galerkin projection
Least-squares Petrov–Galerkin projection
Conservative schemes

ABSTRACT

This work proposes a method for model reduction of finite-volume models that guarantees the resulting reduced-order model is conservative, thereby preserving the structure intrinsic to finite-volume discretizations. The proposed reduced-order models associate with optimization problems characterized by a minimum-residual objective function and nonlinear equality constraints that explicitly enforce conservation over subdomains. Conservative Galerkin projection arises from formulating this optimization problem at the time-continuous level, while conservative least-squares Petrov–Galerkin (LSPG) projection associates with a time-discrete formulation. We equip these approaches with hyper-reduction techniques in the case of nonlinear flux and source terms, and also provide approaches for handling infeasibility. In addition, we perform analyses that include deriving conditions under which conservative Galerkin and conservative LSPG are equivalent, as well as deriving *a posteriori* error bounds. Numerical experiments performed on a parameterized quasi-1D Euler equation demonstrate the ability of the proposed method to ensure not only global conservation, but also significantly lower state-space errors than nonconservative reduced-order models such as standard Galerkin and LSPG projection.

1. Introduction

The finite-volume method is commonly employed for discretizing systems of partial differential equations (PDEs) that associate with conservation laws, especially those in fluid dynamics. Rather than operating on the strong form of the PDE, the finite-volume method operates on the integral form of the PDE to numerically enforce conservation over each control volume comprising the computational mesh. Thus, *conservation* is the primary problem structure imposed by finite-volume discretizations; this contrasts with other discretization techniques that aim to preserve other properties, e.g., variational principles in the case of the finite-element discretizations.

Unfortunately, the computational burden imposed by high-fidelity finite-volume models is often prohibitive, as (1) the fine spatiotemporal resolution typically needed to ensure a verified, validated computational model can lead to extremely

* Corresponding author.
E-mail addresses: ktcarlb@sandia.gov (K. Carlberg), choi15@llnl.gov (Y. Choi), ssusie@uw.edu (S. Sargsyan).
URLs: http://sandia.gov/~ktcarlb (K. Carlberg), http://people.llnl.gov/choi15 (Y. Choi).
[1] Extreme-scale Data Science and Analytics Department, Sandia National Laboratories, Livermore, CA 94550.
[2] This work was performed while employed in the Extreme-scale Data Science and Analytics Department, Sandia National Laboratories, Livermore, CA 94550. *Current affiliation*: Lawrence Livermore National Laboratories. Lawrence Livermore National Laboratory is operated by Lawrence Livermore National Security, LLC, for the U.S. Department of Energy, National Nuclear Security Administration under Contract DE-AC52-07NA27344.
[3] This work was performed while pursing a Ph.D. at the University of Washington during employment as a summer intern in the Extreme-scale Data Science and Analytics Department, Sandia National Laboratories, Livermore, CA 94550. *Current affiliation*: HERE Technologies.

https://doi.org/10.1016/j.jcp.2018.05.019
0021-9991/© 2018 Published by Elsevier Inc.

large-scale models whose simulations consume months on supercomputers, and (2) many engineering problems are *real time* or *many queries* in nature. Such problems require the (parameterized) computational model to be simulated rapidly either due to a strict time-to-solution constraint in the case of real-time problems (e.g., model predictive control) or due to the need for hundreds or thousands of simulations in the case of many-query problems (e.g., statistical inversion).

Reduced-order models (ROMs) have been developed to mitigate this burden. These techniques first perform an *offline* stage during which they execute computationally costly training tasks (e.g., simulating the high-fidelity model for several parameter instances) to compute a low-dimensional 'trial' basis for the state. Next, these methods execute a computationally inexpensive *online* stage during which they rapidly compute approximate solutions for different points in the parameter space by projection: they compute solutions in the span of the trial basis while enforcing the high-fidelity model residual to be orthogonal to the subspace spanned by a low-dimensional 'test' basis. In the presence of nonlinearities, these techniques also introduce 'hyper-reduction' approximations to ensure the cost of simulating the ROM is independent of the high-fidelity-model dimension.

The most popular model-reduction approach for nonlinear dynamical systems such as those arising from finite-volume discretizations is Galerkin projection [52,21,39], wherein the test basis is set to be equal to the trial basis. The trial basis is often computed via proper orthogonal decomposition (POD) [33], but it can also be computed via the reduced-basis method; see Refs. [31,32,30], which apply the classical reduced-basis method to finite-volume problems. More recently, the least-squares Petrov–Galerkin (LSPG) projection method [16,17,15] was proposed, which has been computationally demonstrated to generate accurate and stable responses for turbulent, compressible flow problems on which Galerkin projection yielded unstable responses. Unfortunately, neither Galerkin nor LSPG projection directly preserves important problem structure related to conservation laws or finite-volume models.

To address this, alternative projection techniques have been developed for improving the performance of reduced-order models when applied to conservation laws, particularly those appearing in fluid dynamics. These include stabilizing inner products applied to finite-difference [46] and finite-element discretizations [9,36]; introducing dissipation via closure models [6,51,12,57,49] or numerical dissipation [34]; performing nonlinear Galerkin projection based on approximate inertial manifolds [41,50,35]; including a pressure-term representation [42,28]; modifying the POD basis by including many modes (such that dissipative modes are captured), changing the norm [34], enabling adaptivity [12,14], or including basis functions that resolve a range of scales [7] or respect the attractor's power balance [8]; modifying the projection by adopting a constrained Galerkin [45,26], constrained Petrov–Galerkin [24], or L^1-norm minimizing projection [1]; developing approaches tailored to the incompressible Navier–Stokes equations by introducing stabilizations based on supremizer-enriched velocity spaces and a pressure Poisson equation [54,53] or by modifying the Galerkin projection [38]; and improving the ROM's ability to capture shocks [43,29,14,55]. Among these contributions, only a subset is applicable to finite-volume discretizations. Further, no model-reduction method to date has been developed to preserve the structure intrinsic to finite-volume models: conservation. In particular, none of the above methods ensures that conservation holds over any subset of the computational domain, which can lead to spurious growth or dissipation of quantities that should be conserved in principle.

To this end, this work proposes a novel projection scheme for finite-volume models that ensures the reduced-order model is conservative over subdomains of the problem. The approach leverages the minimum-residual formulation of both Galerkin and least-squares Petrov–Galerkin projection by equipping their associated optimization problems with (generally nonlinear) equality constraints that explicitly enforce conservation over subdomains. The resulting *conservative* reduced-order models can be expressed as the solution to time-dependent saddle-point problems. The approach does not rely on a particular choice of reduced basis, although the reduced basis can affect feasibility of the associated optimization problems. New contributions in this work include:

1. Conservative Galerkin (Section 4.2) and conservative LSPG (Section 4.3) projection techniques, which ensure that the reduced-order models are conservative over subdomains of the original computational mesh. These methods are equipped with
   (a) techniques for handling infeasible constraints (Section 4.4), and
   (b) hyper-reduction techniques that respect the underlying finite-volume discretization to handle nonlinearities in the flux and source terms (Section 4.5).
2. Analysis, which includes:
   (a) demonstration that conservative Galerkin projection and time discretization are commutative (Theorem 4.3),
   (b) sufficient conditions for feasibility of conservative Galerkin (Proposition 5.1) and conservative LSPG (Proposition 5.2) projection,
   (c) conditions under which conservative Galerkin and conservative LSPG projection are equivalent (Theorem 5.1), and
   (d) *a posteriori* bounds (Section 5.3) for the error in the quantities conserved over subdomains (Theorem 5.3), in the null space (Lemma 5.1) and row space (Lemma 5.2) of the constraints, in the full state (Theorem 5.2), and in the conserved quantities (Lemma 5.3 and Theorem 5.3).
3. Numerical experiments on a parameterized quasi-1D Euler equation associated with modeling inviscid compressible flow in a converging–diverging nozzle (Section 6). These experiments demonstrate the merits of the proposed method and illustrate the importance of ensuring reduced-order models are globally conservative.