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# On a Helmholtz transmission problem in planar domains with corners

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ABSTRACT

A particular mix of integral equations and discretization techniques is suggested for the solution of a planar Helmholtz transmission problem with relevance to the study of surface plasmon waves. The transmission problem describes the scattering of a time-harmonic transverse magnetic wave from an infinite dielectric cylinder with complex permittivity and sharp edges. Numerical examples illustrate that the resulting scheme is capable of obtaining total magnetic and electric fields to very high accuracy in the entire computational domain.



## 1. Introduction

This paper is about solving a classic transmission problem for the Helmholtz equation in the plane using integral equation techniques. A physical interpretation is that an incident time-harmonic transverse magnetic wave, in a medium with unit permittivity, is scattered from a homogeneous dielectric cylindrical object with permittivity $\varepsilon$. The problem is to find the total magnetic field $U$ everywhere.

When $\varepsilon$ is real and positive and when the object boundary $\Gamma$ is smooth, this problem is uncomplicated. Efficient boundary integral equations and fast solution techniques have long since been established and their use in computational physics is standard practice. See [26] for pioneering numerical work and [7] for an overview of more recent development. The only issue that, perhaps, still is not completely resolved is how to compute $U$ and its gradient $\nabla U$ in an appropriate fashion close to $\Gamma$ in a post-processor [1,21].

When $\varepsilon$ is not real and positive and when $\Gamma$ is not smooth, the transmission problem gets harder. Issues arise relating to modeling, the existence and the uniqueness of solutions, and resolution. A particularly difficult situation is when $\varepsilon$ is real and negative and $\Gamma$ has sharp corners. The excitation of rapidly oscillating corner fields and their interaction with surface plasmon waves then make the choice of integral equations and discretization techniques crucial. To our knowledge, integral equation methods have not been used in this context, but a finite element solver has recently been developed [5]. This solver, which relies on so-called perfectly matched layers at the corners, is capable of producing convergent results also for challenging setups.

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We will review integral equations for the Helmholtz transmission problem and show that a system of equations due to Kleinman and Martin [20] is well suited for our purposes. In passing we observe that another, seldom used, integral equation from [20] is surprisingly efficient when $\varepsilon$ is real and positive and when the accurate evaluation of $\nabla U$ close to $\Gamma$ is of concern. The successful use of integral equations in computations is, of course, coupled to the choice of discretization scheme. We use standard Nyström discretization, accelerated with recursively compressed inverse preconditioning, and product integration for the evaluation of layer potentials close to their sources [9,11]. As a result, we can solve the transmission problem for negative $\varepsilon$ (in a limit sense) in domains with corners and rapidly obtain corner fields and surface plasmon waves with a precision of about thirteen digits, even close to $\Gamma$.

The rest of the paper is organized as follows: Section 2 presents the transmission problem as a system of partial differential equations (PDEs). Section 3 reviews some popular integral equation reformulations which all work well for $\varepsilon$ real and positive. This includes three systems of integral equations which we call KM0, KM1, and KM2. Section 4 is on discretization. Special emphasis is given to the treatment of singularities and near-singularities of kernels that occur in field representations and systems of integral equations. The basic evaluation strategy is the same as in [9,12], but the treatment of the hypersingularity in the gradient of the acoustic double layer potential operator is new. Section 5 reviews results on the existence and uniqueness of solutions to the PDEs and to the integral equations of KM0, KM1, and KM2. These issues are extremely important when $\varepsilon$ is not real and positive and $\Gamma$ has sharp corners. For inadmissible $\varepsilon$, there simply is no solution. For a discrete set of other $\varepsilon$, an inappropriate choice of integral equations may lead to numerical failure. In Sections 6, 7, and 8 we strive to summarize the fascinating physics which is illustrated by the numerical examples at the end of the paper.

## 2. PDE formulation of the transmission problem

A homogeneous dielectric object, a domain $\Omega_2$ with boundary $\Gamma$, is embedded in a homogeneous dielectric medium $\Omega_1$ in the plane $\mathbb{R}^2$. The outward unit normal at position $r=(x,y)$ on $\Gamma$ is $\nu$. The ratio between the permittivities in $\Omega_2$ and $\Omega_1$ is $\varepsilon$. An incident plane wave

$$U^{\mathrm{in}}(r)=e^{\mathrm{i}k_1(r\cdot d)}\,,\qquad r\in\mathbb{R}^2\,,\tag{1}$$

has wavenumber $k_1$, where $\Re\mathrm{e}\{k_1\}\ge 0$, and direction $d$. Let the wavenumber in $\Omega_2$ be

$$k_2=\sqrt{\varepsilon}k_1\,.\tag{2}$$

A transmission problem for the Helmholtz equation can now be formulated: find $U(r)$ which solves the system of PDEs

$$\Delta U(r)+k_1^2U(r)=0\,,\quad r\in\Omega_1\,,\tag{3}$$

$$\Delta U(r)+k_2^2U(r)=0\,,\quad r\in\Omega_2\,,\tag{4}$$

with boundary conditions

$$\lim_{\Omega_1\ni r\to r^\circ}U(r)=\lim_{\Omega_2\ni r\to r^\circ}U(r)\,,\quad r^\circ\in\Gamma\,,\tag{5}$$

$$\lim_{\Omega_1\ni r\to r^\circ}\varepsilon\nu^\circ\cdot\nabla U(r)=\lim_{\Omega_2\ni r\to r^\circ}\nu^\circ\cdot\nabla U(r)\,,\quad r^\circ\in\Gamma\,,\tag{6}$$

$$U(r)=U^{\mathrm{in}}(r)+U^{\mathrm{sc}}(r)\,,\quad r\in\Omega_1\,,\tag{7}$$

$$U^{\mathrm{sc}}(r)=\frac{e^{\mathrm{i}k|r|}}{\sqrt{|r|}}\left(F(r/|r|)+\mathcal{O}\left(\frac{1}{|r|}\right)\right),\quad |r|\to\infty\,.\tag{8}$$

Here $U^{\mathrm{sc}}(r)$ is the scattered field, $F(r/|r|)$ is the far-field pattern, and (8) is the two-dimensional analogue of the radiation condition [4, Eq. (6.22b)].

We are chiefly interested in computing the real fields

$$H(r,t)=\Re\mathrm{e}\left\{U(r)e^{-\mathrm{i}t}\right\}\,,\qquad r\in\Omega_1\cup\Omega_2\,,\tag{9}$$

$$\nabla H(r,t)=\Re\mathrm{e}\left\{\nabla U(r)e^{-\mathrm{i}t}\right\}\,,\quad r\in\Omega_1\cup\Omega_2\,,\tag{10}$$

where $t$ denotes time and angular frequency is scaled to one. The field $H(r,t)$ can be interpreted as a time-harmonic magnetic wave in a setting where the PDE models a three-dimensional transverse translation-invariant electromagnetic transmission problem for the Maxwell equations, with magnetic and electric fields

$$\boldsymbol{H}(r)=U(r)\hat{\boldsymbol{z}}\,,\tag{11}$$

$$\boldsymbol{E}(r)=\begin{cases}\mathrm{i}k_1^{-1}\nabla U(r)\times\hat{\boldsymbol{z}}\,, & r\in\Omega_1\,,\\ \mathrm{i}k_1^{-1}\varepsilon^{-1}\nabla U(r)\times\hat{\boldsymbol{z}}\,, & r\in\Omega_2\,.\end{cases}\tag{12}$$