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Hierarchical matrix approximation for the uncertainty quantification of potentials on random domains ☆

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ABSTRACT

Computing statistical quantities of interest of the solution of partial differential equations on random domains is an important and challenging task in engineering. We consider the computation of these quantities by the perturbation approach. Especially, we discuss how third order accurate expansions of the mean and the correlation can numerically be computed. These expansions become even fourth order accurate for certain types of boundary variations. The correction terms are given by the solution of correlation equations in the tensor product domain, which can efficiently be computed by means of $\mathcal{H}$-matrices. They have recently been shown to be an efficient tool to solve correlation equations with rough data correlations, that is, with low Sobolev smoothness or small correlation length, in almost linear time. Numerical experiments in three dimensions for higher order ansatz spaces show the feasibility of the proposed algorithm. The application to a non-smooth domain is also included.

1. Introduction

The numerical solution of strongly elliptic linear partial differential equations (PDEs) is an important task in science and engineering. It is nowadays well understood and can be accomplished up to high accuracy, provided that the input data are known exactly. Motivated by tolerances in manufacturing processes and measurement errors, the computation of statistical output functionals of the solution of PDEs on objects with uncertain shapes has recently gained a lot of interest. The domain mapping approach, see, e.g., [1–3], is well suited for modeling large variations in the domain, but usually leads to high-dimensional and costly integration problems, which suffer from the curse of dimensionality. The perturbation approach, see, e.g., [4–7], is motivated by small disturbances in manufacturing processes and models uncertain small deformations under the following view point. Notice that both approaches have recently been combined in [8].

Given a reference domain D_0, the random domains $D_\varepsilon(\omega)$ are defined by the perturbation of the reference domain's boundary in some ε tube, leading to the model problem

$$\begin{aligned} \Delta u_\varepsilon(\omega) &= 0 \quad \text{in } D_\varepsilon(\omega), \\ u_\varepsilon(\omega) &= g \quad \text{on } \partial D_\varepsilon(\omega). \end{aligned}$$

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Using shape calculus, cf. [9,10], and under some smoothness assumptions, the non-linear dependence of the solution $u_\varepsilon(\omega)$ on $D_\varepsilon(\omega)$ can, in a suitable compact subdomain K, be expanded in a Taylor expansion in ε, i.e.,

$$u_\varepsilon(\omega) = u_0 + \varepsilon\delta u(\omega) + \frac{\varepsilon^2}{2}\delta^2 u(\omega) + \mathcal{O}(\varepsilon^3) \quad \text{in } K. \tag{1}$$

The zero order term u_0 can directly be computed by solving the deterministic PDE

$$\begin{aligned} \Delta u_0 &= 0 \quad \text{in } D_0, \\ u_0 &= g \quad \text{on } \partial D_0, \end{aligned}$$

on the unperturbed domain. The first and second order correction terms can be computed by solving the very same equation as for the zero order term, but with different boundary conditions.

Based on the Taylor expansion (1), statistical quantities of the solution like the mean

$$\mathbb{E}[u_\varepsilon] = u_0 + \varepsilon\mathbb{E}[\delta u] + \frac{\varepsilon^2}{2}\mathbb{E}[\delta^2 u] + \mathcal{O}(\varepsilon^3) \quad \text{in } K,$$

the covariance

$$\operatorname{Cov}[u_\varepsilon] = \varepsilon^2 \operatorname{Cor}[\delta u] + \mathcal{O}(\varepsilon^3) \quad \text{in } K \times K,$$

and the correlation

$$\begin{aligned} \operatorname{Cor}[u_\varepsilon] = u_0 \otimes u_0 + \varepsilon\Big(u_0 \otimes \mathbb{E}[\delta u] + \mathbb{E}[\delta u] \otimes u_0\Big) + \varepsilon^2 \operatorname{Cor}[\delta u] \\ + \frac{\varepsilon^2}{2}\Big(u_0 \otimes \mathbb{E}[\delta^2 u] + \mathbb{E}[\delta^2 u] \otimes u_0\Big) + \mathcal{O}(\varepsilon^3) \quad \text{in } K \times K \end{aligned}$$

can be expanded into asymptotic expansions in ε. It has already been shown in [7] that the second order correction term $\operatorname{Cor}[\delta u]$ is the solution to a correlation equation in the higher-dimensional product domain $D_0 \times D_0$. While the first and second order correction terms $\mathbb{E}[\delta u]$ and $\mathbb{E}[\delta^2 u]$ of the mean are given as the solution of PDEs on D_0, the computation of the boundary values for $\mathbb{E}[\delta^2 u]$ has not been investigated yet. We show that they can likewise be obtained by the solution of a correlation equation, but in the product domain $\partial D_0 \times \partial D_0$. As this equation lives solely on the domain boundary, the boundary element method is an obvious discretization method for its solution. Hence, we will use it as a discretization scheme for all occurring equations, omitting the meshing of D_0. We provide the full convergence analysis for the proposed discretization scheme and slightly relax the assumptions from [7] on the boundary perturbations on our way. Additionally, we remark that the asymptotics can even be up to fourth order accurate, if the law of the prescribed boundary variations behaves in a specific way.

We therefore have to solve two correlation equations in the tensor product domain in order to compute third order accurate approximations in the perturbation amplitude ε. As a naive discretization of these higher dimensional problems, also referred to as the full tensor approach, is prohibitively expensive, the solution of such correlation equations has been the topic of several articles, cf. [5,7,11–15] for example. Except for [5,12], where a low-rank approximation of the underlying correlation is employed, all of the mentioned approaches rely in some sense on a sparse tensor approximation. Both, low-rank approximations and sparse tensor discretizations, are best suitable if the prescribed correlation is sufficiently smooth, compare [16,17] for the behavior of low-rank approximations in dependence of the smoothness, and are known to struggle for "rough" correlations. This means that the prescribed correlation exhibits only minor smoothness assumptions or has a high concentration of measure. While rough correlations do not necessarily have an influence on the convergence rates ([18] discusses an specific example where the rate is reduced), they may have a huge influence on the constants involved in the complexity estimates.

Recently, the hierarchical matrix approach (in short $\mathcal{H}$-matrix approach) to correlation equations, cf. [18–20], has been shown to be a promising approach to cope with rough correlations. In the context of correlation equations, $\mathcal{H}$-matrices provide an alternative compression scheme to represent the full tensor product approach and allow the solution of correlation equations in almost linear time. Being introduced in [21,22], $\mathcal{H}$-matrices are feasible for the data-sparse representation of (block-) matrices which can be approximated block-wise with low-rank. They have originally been employed for the efficient treatment of boundary integral equations, as they arise in the boundary element method. Nonetheless, they also provide an arithmetic which can be employed for the solution of matrix equations, as they occur from the discretization of correlation equations.

The rest of this article is organized as follows. In Section 2, we derive expansions of the mean, the covariance, and the correlation with respect to the perturbation's amplitude via shape calculus. Section 3 is concerned with the necessary boundary integral equations to allow for a natural treatment of the random perturbations of the boundary. In Section 4, we introduce the corresponding Galerkin discretization, whereas Section 5 is concerned with its error estimation. Section 6 is concerned with the efficient treatment of the derived equations with $\mathcal{H}$-matrices. We demonstrate the feasibility of the proposed approach by numerical experiments in three spatial dimensions in Section 7. Finally, in Section 8, we draw our conclusions.