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Frequency optimized RBF-FD for wave equations

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ABSTRACT

We present a method to obtain optimal RBF-FD formulas which maximize their frequency range of validity. The optimization is based on the idea of keeping an error of interest (dispersion, phase or group velocity errors) below a given threshold for a wavenumber interval as large as possible. To find the weights of these optimal finite difference formulas we solve an optimization problem. In a previous work we developed a method to optimize the frequency range of validity for finite difference weights. That method required to solve a system of nonlinear equations with as many unknowns as half of the number of weights, which is a very hard task when the number of nodes gets large. The current method requires solving an optimization problem with only one parameter, which makes finding a global minimum easier, and thus can be used for bigger stencils. We also study which of the standard RBF are more appropriate for this problem and introduce a new RBF that depends on two parameters. This new RBF improves the resulting frequency response of the RBF-FD methods while keeping the cost of the optimization problem low.

1. Introduction

Finite-difference (FD) formulas approximate differential operators by a weighted sum of the values of the function at a set of neighboring nodes (stencil). The weights of the standard FD formulas are computed by maximizing the degree of the polynomials for which the FD formula is exact. Classical methods to compute these weights are based on Lagrange's interpolation polynomial, Taylor expansions, or monomial test functions [8]. Similarly, in the case of RBF-FD formulas, the weights are computed by RBF interpolation (instead of polynomial interpolation) on the nodes of the stencil, and by requiring that the formulas are exact for the RBF functions centered at each node of the stencil.

A different approach to design FD formulas is based on their frequency response. Standard formulas are very accurate for low frequencies but they generate large errors at high frequencies. In wave propagation simulations, for example, this leads to dispersive errors, i.e., a progressive distortion of the waveforms as the time and propagation distance increase. To overcome this problem, Holberg [11] proposed in a pioneering work to maximize the range of frequencies for which the group velocity error was bounded within a prescribed value. Thus, the accuracy at low frequencies deteriorates but the range of frequencies correctly modeled increases significantly. Since then, many optimization-based FD formulas can be found in the literature (see [5,13,14] for a review of proposed methods). They differ in the derivative approximated, stencil type and size, objective function, norm minimized, range of frequencies and optimization method used to derive the weights.

In a recent paper [12] we described a method to compute weights of finite difference formulas that maximize the frequency band (or wavenumber range) for which a certain objective function is bounded by a specified value E. We

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considered both conventional and staggered nodes and, in the case of the first derivative, we used as objective function either the dispersion error or the phase velocity error or the group velocity error. We also provided compact formulas to compute the optimal weights as a function of E.

In this paper we analyze the frequency response of RBF-FD formulas. In particular, we compare the frequency response of different RBFs and conclude that Gaussians are best from this point of view. For Gaussians, we also derive compact formulas for the optimal shape parameter (and, therefore, for the optimal weights) that maximizes the spectral frequency band (or wavenumber range) for which a certain objective function is bounded by a specified value E. The frequency band obtained in this way is slightly smaller than the frequency band obtained with optimal weights in [12]. However, the computational cost is much lower. Indeed, with the method proposed here we have to solve an optimization problem for only one variable (the shape parameter), while in [12] we had to solve a nonlinear optimization problem for n variables, where n is the number of weights.

It should be emphasized that the shape parameter derived in this paper is optimal in the sense that it maximizes the spectral frequency band for which the RBF approximation is accurate. Thus, it deteriorates the accuracy at low frequencies in exchange for keeping a reasonable accuracy over a wider range of frequencies. This approach is very different to the optimal shape parameter derived in [1–3] in which the objective was to compute the shape parameter that minimizes the approximation error for a given function.

To our knowledge, there has not been much work on the frequency response of RBF-FD approximations. Notable exceptions are Fornberg and Flyer [6], who used Fourier analysis to study the accuracy of RBF interpolants to trigonometric functions, and Fornberg and Lehto [7] who focused on purely-convective PDEs and how to damp spurious high frequency modes.

The paper is organized as follows: in Section 2 we describe the methodology used to analyze the frequency response of differential operators. In Section 3 we present the main results including the wavenumber dependence of the error, the optimal shape parameter for each of the objective functions considered and some analytic results in the limit of small values of the shape parameter. We also consider the case of two dimensional stencils. In Section 4 we carry out some numerical experiments to highlight the advantages of the proposed RBF-FD weights. Finally in Section 5 we present the main conclusions of the paper.

2. Methodology

The weights of an RBF-FD scheme are calculated solving the linear system

$$A\,\vec{w} = \vec{b}\,. \tag{1}$$

In equation (1) A is the RBF interpolation matrix whose elements are $a_{i,j} = \phi_i(r_j)$, where ϕ_i is an RBF centered at node $\vec{x}_i$ and r_j is the distance between nodes $\vec{x}_i$ and $\vec{x}_j$. $\vec{w}$ is a vector containing the RBF-FD weights, and $\vec{b}$ is a vector whose components are $b_i = L[\phi_i(r_c)]$. Here L is the differential operator we want to approximate and r_c is the distance between node $\vec{x}_i$ and the node where we want to approximate the differential operator. The RBF functions ϕ usually depend on a parameter ϵ, and thus the weights will depend on that parameter too, i.e. $\vec{w} = \vec{w}(\epsilon)$. In the limit that makes the RBF flat ($\epsilon \to 0$), the RBF-FD weights often become (see [8]) the standard finite differences weights.

The procedure is easier to describe on a 1D infinite unit-spaced grid, i.e. $x_j = j$, with j integer. The Fourier modes that are distinguishable from each other on this grid are e^{ikx}, with $|k| \le \pi$. The RBF-FD approximation to the first derivative of these Fourier modes, using a symmetric stencil of $N+1$ nodes (N even) with weights $\vec{w}_j(\epsilon)$, $j = 1, \cdots, N+1$, results in

$$ike^{ikx} \approx 2ie^{ikx} \sum_{j=1}^{N/2} w_j(\epsilon)\,\sin(kj) \equiv i\hat{k}e^{ikx}, \tag{2}$$

where we have used the fact that by symmetry $w_j = -w_{-j}$. The nondimensional effective wavenumber $\hat{k}$ is therefore

$$\hat{k}(k) = 2\sum_{j=1}^{N/2} w_j(\epsilon)\,\sin(kj)\,. \tag{3}$$

Equation (3) gives the numerical approximation $\hat{k}$ to the nondimensional analytic wavenumber k.

Analogously, for a staggered stencil with N (N even) nodes [12]

$$\hat{k} = 2\sum_{j=1}^{N/2} w_j(\epsilon)\,\sin[k\,(2j-1)\,/2]\,. \tag{4}$$

Note that both approximations to the analytical wavenumber (3) and (4) depend on the choice of the weights $w_j(\epsilon)$, $j = 1, \ldots, N/2$, and, therefore, these approximations can be optimized in a wavenumber interval of interest.

This optimization will depend on the objective function to be minimized. For the first derivative we will consider the dispersion error,