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# Hydrodynamic flows on curved surfaces: Spectral numerical methods for radial manifold shapes

B.J. Gross, P.J. Atzberger *,[1]

*Department of Mathematics, Department of Mechanical Engineering, University of California, Santa Barbara, United States*

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ABSTRACT

We formulate hydrodynamic equations and spectrally accurate numerical methods for investigating the role of geometry in flows within two-dimensional fluid interfaces. To achieve numerical approximations having high precision and level of symmetry for radial manifold shapes, we develop spectral Galerkin methods based on hyperinterpolation with Lebedev quadratures for $L^2$-projection to spherical harmonics. We demonstrate our methods by investigating hydrodynamic responses as the surface geometry is varied. Relative to the case of a sphere, we find significant changes can occur in the observed hydrodynamic flow responses as exhibited by quantitative and topological transitions in the structure of the flow. We present numerical results based on the Rayleigh-Dissipation principle to gain further insights into these flow responses. We investigate the roles played by the geometry especially concerning the positive and negative Gaussian curvature of the interface. We provide general approaches for taking geometric effects into account for investigations of hydrodynamic phenomena within curved fluid interfaces.



## 1. Introduction

We develop spectral numerical methods for a continuum mechanics formulation of hydrodynamic flows within two-dimensional curved fluid interfaces. Hydrodynamics within curved geometries play an important role in diverse physical systems including the thin films of soap bubbles [30,46,48,49], lipid bilayer membranes [43,74,77,83,92] and recent interface-embedded colloidal systems [16,19,20,29,78]. Similar hydrodynamic and related curvature mediated phenomena also plays an important role in physiology such as in the cornea of the eye with its tear film [14], transport of surfactants in lung alveoli [40,60] or in cell mechanics [67,70,74]. Each of these systems involve potential interactions between the curvature of the interface and hydrodynamic flows. We investigate these types of flows by formulating continuum mechanics equations for hydrodynamics using variational principles and the exterior calculus of differential geometry [61]. This provides an abstraction that is helpful in generalizing many of the techniques of fluid mechanics to the manifold setting while avoiding many of the tedious coordinate-based calculations of tensor calculus. The exterior calculus formulation also provides a coordinate-invariant set of equations helpful in providing insights into the roles played by the geometry in the hydrodynamics.

There has been a significant amount of experimental and theoretical work developing approaches for investigating hydrodynamics within curved fluid interfaces [50,73,75,88]. Experimental work includes single particle tracking of inclusions to

* Corresponding author.
*E-mail address:* atzberg@gmail.com (P.J. Atzberger).
*URL:* http://atzberger.org/ (P.J. Atzberger).

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determine information about interfacial viscosity and diffusivities [77,105]. Even the formulation of the correct continuum mechanics equations presents some significant challenges in the manifold setting [61]. For instance in a two-dimensional curved fluid sheet the equations must account for the distinct components of linear momentum correctly. The concept of momentum is not an intrinsic field of the manifold and must be interpreted with respect to the ambient physical space [61]. For instance, when considering non-relativistic mechanics in an inertial reference frame with coordinates $x, y, z$, the $x$-component of momentum is a conserved quantity distinct from the $y$ and $z$-components of momentum. An early derivation using coordinate-based tensor calculus in the ambient space was given for hydrodynamics within a curved two-dimensional fluid interface in Scriven [88]. This was based on the more general shell theories developed in [28,109]. Many subsequent derivations have been performed using tensor calculus for related fluid–elastic interfaces motivated by applications. This includes derivation of equations for surface rheology [25,63,64], investigation of red-blood cells [89], surface transport in capsules and surfactants on bubbles [30,75], and investigations of the mechanics, diffusion, and fluctuations associated with curved lipid bilayer membranes [18,23,39,79,80,84,92,95,100]. Recent works by Marsden et al. [45,61,108] develop the continuum mechanics in the more general setting when both the reference body and ambient space are treated as general manifolds as the basis for rigorous foundations for elasticity [45,61]. In this work, some of the challenges associated with momentum and stress with reference to ambient space can be further abstracted in calculations by the use of covector-valued differential forms and a generalized mixed type of divergence operator [45,108]. A particularly appealing way to derive the conservation laws for manifolds is through the use of variational principles based on the balance of energy and symmetries [61]. This has recently been pursued to derive elastic and hydrodynamic equations for lipid membranes in [5,36,84]. We briefly present related derivations based on the energy balance approach of [61,108] to obtain our hydrodynamic equations in Section 2.

There has been a lot of recent interest and work on developing computational methods for evaluating differential operators and for solving equations on curved surfaces [27,31,34,87]. This has been motivated in part by applications in computer graphics [9,22,31] and interest in applications using shell theories for elasticity and hydrodynamics [5,23,36,69,74,86,107]. Many computational methods treat the geometry using a triangulated surface and build discrete operators to model their curvature and differential counter-parts [58,62,65,96,98,106,110]. Some early work in this direction includes [58,65,96,98] and the Surface Evolver of Brakke [13]. More recently, discrete approaches such as the Discrete Exterior Calculus (DEC) [26,62], Finite Element Exterior Calculus (FEEC) [1,4] and Mimetic Methods (MM) [12] have been developed that aim to reproduce in the numerics analogous properties of the differential operators related to the geometric and topological structure of the manifold [1,4,12,26,42,62,110]. For manifolds represented by discrete symplicial complexes this includes preserving the adjoint conditions between the exterior derivative, boundary operator, and co-differentials to create a discrete analogue of the de Rham complex and related theory [2,44,94]. This has been used to obtain models of surface Laplacians and results like discrete Hodge decompositions [1,4,12,31]. For finite elements these properties can be shown to be essential for discretizing problems in elasticity and fluid mechanics to obtain well-posed approximations with stable numerical methods [1,4,12]. In the DEC approach to formulating numerical methods for PDEs on manifolds, the methods obtained are similar to finite differences [42,62]. This work has allowed for impressive results including schemes that are exactly conservative for quantities such as mass and vorticity [27,68]. Deriving operators preserving geometric structure is non-trivial and current numerical methods for spherical topologies are typically first or second order accurate [27,68]. Recent methods have been developed in the setting of tensor product basis that are spectrally accurate in [81].

Here, we develop spectrally accurate methods for solving hydrodynamic flows on curved surfaces having general radial manifold shape based on an exterior calculus formulation of the hydrodynamic equations. While our numerical methods do not seek to preserve exactly the geometric and topological relations between our approximate exterior operators, we have from the spectral representation that these relations hold for our expansions to a high level of accuracy. In our derivations we make use of the relations in the exterior calculus such as the Hodge decomposition and adjoint conditions on exterior derivatives and co-differentials to obtain our weak approximations to the hydrodynamic equations.

We develop spectral numerical methods based on Galerkin approximations with hyperinterpolation for $L^2$-projection to spherical harmonics based on Lebedev quadrature. The Lebedev quadrature nodes are derived by solving a non-linear system of equations that impose both exactness of integration on spherical harmonics up to a specified order while maintaining symmetry under octahedral rotations and reflections [54,55]. While one could also consider using a quadrature based on spherical coordinates and sampling on the latitudinal and longitudinal points which have some computational advantages by using the Fast Fourier Transform [24,38,51], as we discuss in Section 3.1.1, these nodes have significant asymmetries with nodes concentrated in clusters near the poles of the sphere. Since Lebedev nodes were developed for quadratures on the sphere, we extend them to obtain quadratures for general radial manifolds by making use of a coordinate-independent change of measure formula derived using the Radon–Nikodym Theorem [57]. We mention that alternative formulations are also possible related to our approach in terms of discrete triangulations provided appropriate transport theorems and quadrature are developed over the mesh. We test the accuracy of our quadrature scheme for general radial manifolds by integrating the Gaussian curvature over the surface. From the Gauss–Bonnet Theorem this should give the Euler characteristic for the spherical topology and independent of the detailed geometric shape [76,94]. We then provide convergence results for our hydrodynamics solver in a few special cases with known hydrodynamics solutions showing the solver's accuracy.

We demonstrate our numerical methods for a few example manifolds by investigating hydrodynamic flow responses and the role of surface geometry. As a baseline, we first consider hydrodynamic flows driven by particles configured on a sphere and subjected to force. We investigate for equivalent forcing how these hydrodynamic flow responses change when