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Dispersion analysis of compatible Galerkin schemes for the 1D shallow water model

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ABSTRACT

In this work, we study the dispersion properties of two compatible Galerkin schemes for the 1D linearized shallow water equations: the $P_n^C - P_{n-1}^{DG}$ and the $GD_n - DGD_{n-1}$ element pairs. P_n is the order n Lagrange space, P_{n-1}^{DG} is the order $n-1$ discontinuous Lagrange space, GD_n is the order n Galerkin difference space, and DGD_{n-1} is the order $n-1$ discontinuous Galerkin difference space. Compatible Galerkin methods have many desirable properties, including energy conservation, steady geostrophic modes and the absence of spurious stationary modes, such as pressure modes. However, this does not guarantee good wave dispersion properties. Previous work on the $P_2^C - P_1^{DG}$ pair has indeed indicated the presence of spectral gaps, and it is extended in this paper to the study of the $P_n^C - P_{n-1}^{DG}$ pair for arbitrary n. Additionally, an alternative element pair is introduced, the $GD_n - DGD_{n-1}$ pair, that is free of spectral gaps while benefiting from the desirable properties of compatible elements. Asymptotic convergence rates are established for both element pairs, including the use of inexact quadrature (which diagonalizes the velocity mass matrix) for the $P_n^C - P_{n-1}^{DG}$ pair and reduced quadrature for the $GD_n - DGD_{n-1}$ pair. Plots of the dispersion relationship and group velocities for a wide range of n and Rossby radii are shown. A brief investigation into the utility of mass lumping to remove the spectral gaps for the $P_3^C - P_2^{DG}$ pair is performed. Finally, a pair of numerical simulations are run to investigate the consequences of the spectral gaps and highlight the main differences between the two elements.

1. Introduction

The application of compact Galerkin methods for numerical models of geophysical fluid flows has become increasingly common over the last 15 years. Of particular interest are compatible finite element methods [1,2] or their closely related generalization compatible Galerkin methods [3]. Compatible Galerkin methods are the generalization of discrete deRham complexes from finite element exterior calculus [4] to more general compact Galerkin methods that are not finite elements. Some compatible finite element pairs were first investigated in [5,6]. Such pairs represent the extension of Arakawa C-grid finite difference schemes (also known as staggered grid method or Marker and Cell methods) to a Galerkin approach. Due to this, they have many desirable properties, such as energy conservation, steady geostrophic modes and various mimetic properties. Additionally, compatible Galerkin methods do not suffer from spurious pressure modes or inertial modes, although all

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known examples do have a CD/Coriolis mode [1,7] due to the discrete Coriolis matrix being rank-deficient. However, even then compatible Galerkin schemes are not guaranteed to have good wave dispersion properties. Further care in the choice of spaces is required to ensure a lack of spurious branches and the absence of spectral gaps in the dispersion relationship. A precise definition for branches, spurious branches and spectral gaps will be given in Section 3. The dispersion properties of compatible Galerkin schemes for the 1D linear shallow water equations is the focus of this paper.

The history of dispersion analysis goes back to the dawn of geophysical fluid modeling [8]. However, only recently have the dispersion properties of compatible finite element methods for the shallow water model been investigated. The $RT_0 - P_0^{DG}$ and/or $BDM_1 - P_0^{DG}$ pairs on triangles were studied in [5–7,9,10]. These are the lowest-order members of the corresponding $P_r^- \Lambda^k$ (RT_0) and $P_r \Lambda^k$ (BDM_1) families from finite element exterior calculus [4]. Unfortunately both elements have spurious branches of the dispersion relationship, and the presence or absence of spectral gaps is still unclear. Such gaps are unphysical numerical artifacts, and are a general feature of high order finite element discretizations [11–13]. The compound finite elements (which are a compatible finite element method on arbitrary polygons) introduced in [14] were studied in [15] for quadrilaterals and hexagons, and found to be quite similar to the corresponding C-grid finite difference schemes. The $P_2^C - P_1^{DG}$ pair in 1D was analyzed in [16], and a spectral gap was found. A solution to this gap for the nonrotating linear shallow water equations, obtained through partial lumping of the velocity mass matrix, is given in [17]. This approach was extended to the $RT_1 - P_1^{DG}$ pair on quadrilaterals for the 2D rotating linear shallow water equations in [18]. When this work was started, detailed study of the dispersion properties of the $P_n^C - P_{n-1}^{DG}$ pair for arbitrary n was still lacking, and was one of the focuses of this work. However, late in the process of preparing this manuscript, we became aware of a related, independent study for the $P_n^C - P_{n-1}^{DG}$ element [19], which also studied the 2D dispersion relationship for pure gravity waves. This work extends [19] in several keys ways: a focus on inertia-gravity waves rather than the limiting cases of pure gravity ($f = 0$) or pure inertia ($g = H = 0$) waves; determination of the asymptotics for the $P_n^C - P_{n-1}^{DG}$ pair with both exact and inexact quadrature, and also for the partially mass lumped version from [17,18]; and an alternative branch selection procedure (see Section 3.5).

In this paper, the dispersion properties of two compatible Galerkin schemes are studied for the 1D linearized shallow water equations: the $P_n^C - P_{n-1}^{DG}$ and the $GD_n - DGD_{n-1}$ finite element pairs. P_n is the order n Lagrange space, P_{n-1}^{DG} is the order $n-1$ discontinuous Lagrange space, GD_n is the order n Galerkin difference space, and DGD_{n-1} is the order $n-1$ discontinuous Galerkin difference space. More details on these two pairs are provided in Section 4 and Section 5. We show that the number and width of the spectral gaps for the $P_n^C - P_{n-1}^{DG}$ pair increases as n increases. The $P_n^C - P_{n-1}^{DG}$ pair is investigated for both exact and inexact quadrature, with inexact leading to a diagonal velocity mass matrix. In addition, partial lumping of the velocity mass matrix is explored as a means to eliminate the spectral gaps that occur for $n \geq 2$. The presence of such gaps motivates the introduction of the $GD_n - DGD_{n-1}$ pair that does not suffer from spectral gaps. For the $GD_n - DGD_{n-1}$ pair, both exact and reduced quadrature are investigated, with reduced quadrature ameliorating some of the computational cost, although the mass matrix remains non-diagonal. Finally, both schemes are compared using a range of n for two test cases, and some conclusions about their applicability to the development of a full geophysical fluid model are drawn. This work represents a starting point for the analysis of the dispersion properties of compatible Galerkin schemes applied to the shallow water equations: in particular, an extension to 2D inertia-gravity waves and the incorporation of time discretization remains to be done.

2. Model problem

Consider the 1D, inviscid shallow water (SW) equations with constant Coriolis parameter f and a flat bottom, linearized about a state of rest with constant fluid depth H. Such a formulation is satisfactory for our purpose, which in Cartesian coordinates is expressed [20] as

$$\frac{\partial u}{\partial t} - f\, v + g\,\frac{\partial \eta}{\partial x} = 0\,, \tag{1}$$

$$\frac{\partial v}{\partial t} + f\, u = 0\,, \tag{2}$$

$$\frac{\partial \eta}{\partial t} + H\,\frac{\partial u}{\partial x} = 0\,, \tag{3}$$

where $\mathbf{u}(x,t) = (u,v)$ is the velocity component, $\eta(x,t)$ is the surface elevation with respect to the reference level $z = 0$ and g is the gravitational acceleration. Note that η would be the pressure in the Navier–Stokes equations. Equations (1)–(3) describe a first order hyperbolic system, and initial conditions and periodic boundary conditions (which are employed for the subsequent Fourier analyses) complete the mathematical statement of the problem.

The linear stability of (1)–(3) is first examined. Because (1)–(3) form a linear system with constant coefficients in a periodic domain, it is sufficient to consider a single Fourier mode, and the variables can be written as $\eta = \hat{\eta} e^{-i\omega t} e^{ikx}$, $u = \hat{u} e^{-i\omega t} e^{ikx}$ and $v = \hat{v} e^{-i\omega t} e^{ikx}$, where $\hat{\eta}$, $\hat{u}$ and $\hat{v}$ are the Fourier amplitudes, ω is the temporal frequency and k is the wavenumber. Substitution into (1)–(3) leads to

$$-i\omega\hat{u} - f\hat{v} + ikg\hat{\eta} = 0, \tag{4}$$