Journal of Computational Physics 370 (2018) 58–84

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Assessment of Hybrid High-Order methods on curved meshes and comparison with discontinuous Galerkin methods ☆

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A R T I C L E I N F O

Article history:
Received 5 September 2017
Received in revised form 2 May 2018
Accepted 9 May 2018
Available online xxxx

Keywords:
Hybrid High-Order methods
Discontinuous Galerkin methods
Curved meshes
Curved elements
Poisson problem

A B S T R A C T

We propose and validate a novel extension of Hybrid High-Order (HHO) methods to meshes featuring curved elements. HHO methods are based on discrete unknowns that are broken polynomials on the mesh and its skeleton. We propose here the use of physical frame polynomials over mesh elements and reference frame polynomials over mesh faces. With this choice, the degree of face unknowns must be suitably selected in order to recover on curved meshes the same convergence rates as on straight meshes. We provide an estimate of the optimal face polynomial degree depending on the element polynomial degree and on the so-called effective mapping order. The estimate is numerically validated through specifically crafted numerical tests. All test cases are conducted considering two- and three-dimensional pure diffusion problems, and include comparisons with discontinuous Galerkin discretizations. The extension to agglomerated meshes with curved boundaries is also considered.

1. Introduction

The continuous growth of high-performance computational resources and the increasing predictive capabilities of numerical models has significantly widened the range of real-life, multi-physics configurations that can be simulated. The trend is towards increasingly complex systems of partial differential equations (PDEs) in complex domains, possibly focusing on a multiscale spatial and temporal behavior. Also the amount of physical data (permeability, mechanical properties, etc.) to be incorporated into large-scale models in order to replicate the complexity encountered in the real-world is rapidly increasing. In this context, the geometrical flexibility of numerical methods is a crucial aspect that can greatly reduce the effort required to obtain an accurate representation of both the computational domain and the problem data. We develop and numerically investigate here a specific instance of discretization methods for PDEs that support high-order approximation on curved (high-order) meshes. Curved meshes are commonly employed to provide a satisfactory representation of the domain boundary with only a moderate number of mesh elements, so that the polynomial degree can be increased while keeping the global number of degrees of freedom (DOFs) under control. The role that curved meshes play in obtaining accurate solutions when combined with high-order discretization methods has been demonstrated, e.g., in [1–3].

☆ We acknowledge the support of Agence Nationale de la Recherche through grant HHOMM (ANR-15-CE40-0005) and of University of Bergamo through the project ITALY – Italian TALented Young researchers.

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https://doi.org/10.1016/j.jcp.2018.05.017
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In recent years, discretization methods supporting arbitrary approximation orders on general meshes have received an increasing amount of attention. We cite here, among others, Discontinuous Galerkin (DG) and Hybridizable Discontinuous Galerkin (HDG) methods, see e.g. [4–7], Hybrid High-Order (HHO) methods [8,9], and Virtual Element Methods (VEM) [10]. The implementation of efficient DG and HDG methods on curved meshes is an open field of research. On the one hand, both Bassi et al. [5] and Warburton [11] proposed the use of polynomial spaces defined in the physical frame. In the former reference, orthonormal bases are obtained by means of a modified Gram–Schmidt procedure to ensure numerical-stability at high-polynomial degrees, while in the latter the same goal is attained by incorporating the spatial variation of the element Jacobian into the physical basis functions. On the other hand, recent works by Chan et al. [12,13] rely on reference frame polynomial spaces introducing weight-adjusted L^2-inner products in order to recover high-order accuracy. HDG has been employed on meshes with curved boundaries, mainly in the context of compressible flow problems [14,15]; eXtended HDG with level-set description of interfaces has been recently investigated by Gurkan et al. [16]. Fidkowski [17] compared DG and HDG methods for unsteady simulations of convection-dominated flows on mapped deforming domains. Blended isogeometric DG methods formulated on elements that exactly preserve the CAD geometry have also been recently proposed in [18]. Finally, we cite here the very recent work [19] of Beirão da Veiga et al. on two-dimensional Virtual Element methods supporting meshes with curved edges.

To this day, HHO methods have been essentially confined to meshes with straight edges in two space dimensions and planar faces in three space dimensions. In this work, we devise a novel extension of HHO methods to meshes featuring curved elements, assess its performance, and compare it with DG methods. HHO methods are based on discrete unknowns that are broken polynomials on elements and faces, and rely on two key ingredients: (i) local reconstructions obtained by solving small, embarrassingly parallel problems inside each element and (ii) high-order stabilization terms penalizing face residuals. These ingredients are combined to formulate local contributions, which are then assembled as in standard finite elements. The construction is devised so that only face unknowns are globally coupled (element unknowns can be locally eliminated by static condensation), leading to global problems of relatively small size and compact stencil that can be solved efficiently, both sequentially and in parallel.

The crucial issue to extend HHO methods to curved meshes lies in the definition of face unknowns, for which we propose the use of reference frame polynomials. With this choice, the degree of face unknowns must be suitably selected in order to recover on curved meshes the same convergence rates as on straight meshes. We provide an estimate of the optimal face polynomial degree depending on the element polynomial degree and on the so-called effective mapping order; see (23) below. The performance of the resulting method applied to a pure diffusion problem is thoroughly assessed through a comprehensive set of tests. Specifically, the numerical results presented in Section 5 compare h- and p-convergence rates of HHO and DG methods over two- and three-dimensional curved meshes. We consider both randomly and regularly distorted mesh sequences, which do not tend to affine meshes upon refinement, as well element subdivision mesh sequences, where mesh elements have faces that are less and less curved (asymptotically affine elements). In Section 6, we also consider p-convergence on curved computational domains discretized by means of agglomerated meshes in the spirit of [5].

The material is organized as follows. In Section 2 we introduce the discrete setting (mesh, mapping functions, and numerical integration). In Section 3 we discuss local polynomial spaces over elements and faces and projections thereon. The HHO and DG discretizations of the Poisson problem used for the numerical study are formulated in Section 4. The numerical results on standard curved and agglomerated meshes are collected in Sections 5 and 6, respectively. Finally, some conclusions are drawn in Section 7.

2. Discrete setting

In this section we discuss the main assumptions on the mesh and provide details on the functions that realize the mapping from reference geometries to physical elements, as well as on the numerical computation of integrals over elements and faces.

2.1. Mesh

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded connected open domain with Lipschitz boundary. For any $n \in \{1,\dots,d\}$, let $\mathcal{K}^n$ be a fixed set of reference geometries defined in the Cartesian frame $\boldsymbol{\xi} = \{\xi_i\}_{1\leqslant i\leqslant n}$. The set of reference geometries contains, e.g., triangular and quadrilateral reference elements for $n=2$, tetrahedral, hexahedral, pyramidal and prismatic reference elements for $n=3$.

We consider a possibly curved mesh $\mathcal{T}_h$ of Ω in the usual finite element sense, i.e., $\mathcal{T}_h$ is a set of disjoint open elements $T \in \mathcal{T}_h$ with non-empty interior that satisfy

$$\overline{\Omega} = \sum_{T\in\mathcal{T}_h} \overline{T}, \tag{1}$$

and it holds that $h = \max_{T\in\mathcal{T}_h} h_T$ with h_T denoting the diameter of T. Notice that (1) entails a simplification: more generally, $\overline{\Omega}$ is only approached as the meshsize h tends to 0, as is the case for the numerical tests of Section 6. Mesh faces are collected in the set $\mathcal{F}_h$, partitioned as $\mathcal{F}_h = \mathcal{F}_h^{\rm i} \cup \mathcal{F}_h^{\rm b}$, where $\mathcal{F}_h^{\rm i}$ collects internal faces and $\mathcal{F}_h^{\rm b}$ boundary faces. For any mesh element $T \in \mathcal{T}_h$, the set $\mathcal{F}_T := \{F \in \mathcal{F}_h : F \subset \partial T\}$ collects the mesh faces composing the boundary of T.