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Numerical solution of an inverse boundary value problem for the heat equation with unknown inclusions

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ABSTRACT

We consider the problem of reconstructing unknown inclusions inside a thermal conductor from boundary measurements, which arises from active thermography and is formulated as an inverse boundary value problem for the heat equation. In our previous works, we proposed a sampling-type method for reconstructing the boundary of the unknown inclusion and gave its rigorous mathematical justification. In this paper, we continue our previous works and provide a further investigation of the reconstruction method from both the theoretical and numerical points of view. First, we analyze the solvability of the Neumann-to-Dirichlet map gap equation and establish a relation of its solution to the Green function of an interior transmission problem for the inclusion. This naturally provides a way of computing this Green function from the Neumann-to-Dirichlet map. Our new findings reveal the essence of the reconstruction method. A convergence result for noisy measurement data is also proved. Second, based on the heat layer potential argument, we perform a numerical implementation of the reconstruction method for the homogeneous inclusion case. Numerical results are presented to show the efficiency and stability of the proposed method.

1. Introduction

Consider the heat conduction in a two-layered medium (Fig. 1.1). Denote by D_0 and D the outer and inner layers, respectively. Set $\Omega = \overline{D} \cup D_0$. Suppose that the thermal conductivities of D_0 and D are 1 and k, respectively. We also assume that the boundaries ∂D_0 and ∂D of D_0 and D, respectively, are of class C^2. For simplicity of notations, throughout this paper we denote $X \times (0, T)$ and $\partial X \times (0, T)$ by X_T and $(\partial X)_T$, respectively, where X is a bounded domain in $\mathbb{R}^2$ and ∂X denotes its boundary. Injecting a heat flux g on $\partial\Omega$ over some time interval $(0, T)$, the temperature distribution u in Ω_T can be modeled by the following initial-boundary value problem:

$$\begin{cases} (\partial_t - \nabla \cdot k\nabla)u = 0 & \text{in } D_T, \\ (\partial_t - \Delta)u = 0 & \text{in } (\Omega \setminus \overline{D})_T, \\ u|_- - u|_+ = 0 & \text{on } (\partial D)_T, \\ k\partial_\nu u|_- - \partial_\nu u|_+ = 0 & \text{on } (\partial D)_T, \\ \partial_\nu u = g & \text{on } (\partial\Omega)_T, \\ u = 0 & \text{at } t = 0, \end{cases} \tag{1.1}$$

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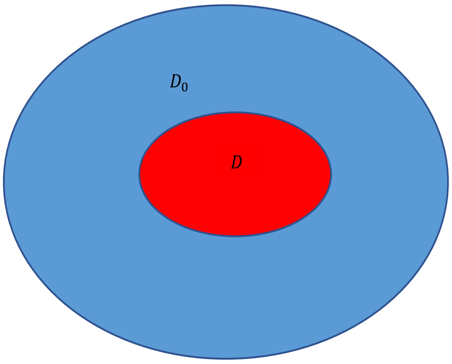

Fig. 1.1. Configuration of the medium.

where ν on ∂D (or $\partial\Omega$) is the unit normal vector directed into the exterior of D (or Ω). Here the subscripts "+" and "−" indicate the trace taken from the exterior and interior of D, respectively.

The above model has many important applications in sciences and engineering. In active thermography, D is regarded as an inclusion and D_0 is the background medium. In this case, the forward problem is to determine the temperature distribution in Ω_T for any injected heat flux g on $(\partial\Omega)_T$, while the inverse problem is to reconstruct the unknown inclusion D from boundary measurements. Instead of recovering the thermal conductivity, we are more interested in finding the location, size and shape of the inclusion as a defect inside the conductor. We proved in [17] that for any $g \in H^{-\frac{1}{2},-\frac{1}{4}}((\partial\Omega)_T)$ there exists a unique solution u to (1.1) in $\tilde{H}^{1,\frac{1}{2}}(\Omega_T)$. Define the Neumann-to-Dirichlet map by

$$\Lambda_D : H^{-\frac{1}{2},-\frac{1}{4}}((\partial\Omega)_T) \to H^{\frac{1}{2},\frac{1}{4}}((\partial\Omega)_T), \quad g \mapsto u|_{(\partial\Omega)_T},$$

which is an idealized measurement data for active thermography. Then our inverse problem is to reconstruct D from Λ_D, where k is unknown. The uniqueness and stability estimate are established in [7,8]. As for reconstruction methods, we refer to [6,9,13–16,21] and the references therein, where the dynamical probe method and the enclosure method are developed. Recently, the authors established a linear sampling-type method for the heat equation in [12,17,18]. However, numerical studies of these reconstruction methods for parabolic inverse boundary value problems are rather limited [19]. Some related works on other kinds of parabolic inverse boundary value problems can be found in [2,3,5,10,11,20].

In this work, based on the heat layer potential theory, we investigate both the forward and inverse problems from the numerical point of view. Especially, the sampling-type reconstruction method established in [17] for our inverse problem will be numerically implemented. Roughly speaking, this reconstruction method is based on the characterization of the solution to the so-called Neumann-to-Dirichlet map gap equation

$$(\Lambda_D - \Lambda_\emptyset)g = G^{\Omega}_{(y,s)}(x,t), \tag{1.2}$$

where $\Lambda_\emptyset$ is the Neumann-to-Dirichlet map when $D = \emptyset$, and $G^{\Omega}_{(y,s)}(x,t) := G^{\Omega}(x,t;y,s)$ is the Green function for the heat operator $\partial_t - \Delta$ in Ω_T with homogeneous Neumann boundary condition on $(\partial\Omega)_T$. In terms of this characterization, the norm of the solution to (1.2) serves as an indicator function and the boundary of D can be reconstructed approximately by computing the values of the indicator function at a set of sampling points. Although the sampling-type reconstruction method for inverse scattering problems has been extensively studied; see [1] and the references therein, very few numerical results for parabolic inverse boundary value problems are reported. We recently studied in [19] the numerical implementation of the sampling method for identifying unknown cavities in the thermal conductor, but the rigid inclusion case has not yet considered.

In this paper, we continue our previous works and investigate the numerical realization of the sampling method for parabolic inverse boundary value problems with unknown inclusions. First of all, we supplement the theoretical analysis of our reconstruction method by analyzing the solvability of the equation (1.2) and showing the relation of its solution to the Green function of an associated interior transmission problem. These new findings reveal the essence of the sampling-type reconstruction method. In addition, a convergence result for noisy measurement data is proved. Then, we simulate the measurement data Λ_D by solving the forward problem (1.1), and compute the Neumann-to-Dirichlet map $\Lambda_\emptyset$ and the Green function $G^{\Omega}_{(y,s)}(x,t)$ by solving the problem (1.1) with $D = \emptyset$. By expressing the solution as a single-layer heat potential, the initial-boundary value problem (1.1) is transformed into a system of boundary integral equations. A numerical scheme for solving the resulting integral equations is introduced. Finally, we solve the discretized Neumann-to-Dirichlet map gap equation using the Tikhonov regularization technique. We show the performance of the reconstruction method from the following two aspects. 1. We test the method for inclusions of different shapes and thermal conductivities. 2. We test the method with short time measurements, namely, using measured data only in a very short time interval. Our numerical results illustrate the efficiency of the reconstruction method.