Journal of Computational Physics 369 (2018) 103–124

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Reconstructed discontinuous Galerkin methods for linear advection–diffusion equations based on first-order hyperbolic system

Jialin Lou [a,1], Lingquan Li [a,1], Hong Luo [a,*,2], Hiroaki Nishikawa [b,3]

[a] Department of Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, NC 27695, USA
[b] National Institute of Aerospace, Hampton, VA 23666, USA

ARTICLE INFO

Article history:
Received 17 October 2017
Received in revised form 26 April 2018
Accepted 28 April 2018
Available online 8 May 2018

Keywords:
Reconstructed Discontinuous Galerkin methods
First-order hyperbolic system
Advection–diffusion equations
Unstructured grids

ABSTRACT

Newly developed reconstructed Discontinuous Galerkin (rDG) methods are presented for solving linear advection–diffusion equations on hybrid unstructured grids based on a first-order hyperbolic system (FOHS) formulation. Benefiting from both FOHS and rDG methods, the developed hyperbolic rDG methods are reliable, accurate, efficient, and robust, achieving higher orders of accuracy than conventional DG methods for the same number of degrees-of-freedom. Superior accuracy is achieved by reconstruction of higher-order terms in the solution polynomial via gradient variables introduced to form a hyperbolic diffusion system and least-squares/variational reconstruction. Unsteady capability is demonstrated by an L-stable implicit time-integration scheme. A number of advection–diffusion test cases with a wide range of Reynolds numbers, including boundary layer type problems and unsteady cases, are presented to assess accuracy and performance of the newly developed hyperbolic rDG methods. Numerical experiments demonstrate that the hyperbolic rDG methods are able to achieve the designed optimal order of accuracy for both solutions and their derivatives on regular, irregular, and heterogeneous grids, indicating that the developed hyperbolic rDG methods provide an attractive and probably an even superior alternative for solving the linear advection–diffusion equations.

1. Introduction

Nowadays, the discontinuous Galerkin (DG) methods, originally developed for solving the steady neutron transport [1] and unsteady advection problems [2], have shown increasing attention in science and engineering field for solving conservation laws. They are widely used in computational fluid dynamics (CFD), computational acoustics, and computational magneto-hydrodynamics. By combining the advantages of the finite element (FE) and finite volume (FV), DG methods, one can achieve high order accuracy while retaining the compactness of the stencil. Meanwhile, DG methods are especially suitable for hyperbolic-type systems of equations in terms of solution accuracy [3–7], treatment of non-conforming meshes [8], and implementation of the hp-adaptivity [9]. However, the DG methods have a number of their own weaknesses. In partic-

* Corresponding author.
E-mail address: hong_luo@ncsu.edu (H. Luo).
[1] PhD Student.
[2] Professor.
[3] Associate Research Fellow.

https://doi.org/10.1016/j.jcp.2018.04.058
0021-9991/

ular, how to reduce the computing costs for the DG methods, and how to discretize and efficiently solve elliptic/parabolic equations remain two unresolved and challenging issues in the DG methods.

In order to reduce both computational costs and storage requirements of DG methods, a new family of reconstructed DG methods, termed P_nP_m schemes, referred to as rDG(P_nP_m) in this paper, was introduced by Dumbser et al. [10–12]. Here, P_n indicates that a piecewise polynomial of degree of n is used to represent an underlying DG solution, and P_m represents a reconstructed polynomial solution of degree of m ($m \geq n$) that is used to compute the fluxes and source terms. Note that the rDG(P_nP_m) schemes provide a unified formulation for both FV and DG methods, and contain both classical FV and standard DG methods as two special cases of rDG(P_nP_m) schemes. Obviously, the construction of an accurate and efficient reconstruction operator is crucial to the success of the rDG(PnPm) schemes. In Dumbser's work [10–12], a higher order polynomial solution is reconstructed using a L2 projection, requiring it indistinguishable from the underlying DG solutions in the contributing cells in the weak sense. The resultant over-determined system is then solved using a least-squares method that guarantees exact conservation, not only of the cell averages but also of all higher order moments in the reconstructed cell itself, such as slopes and curvatures. However, this conservative least-squares reconstruction approach is computationally expensive, as the L2 projection, i.e., the operation of integration, is required to obtain the resulting over-determined system. Furthermore, the reconstruction might be problematic for a boundary cell, where the number of the face-neighboring cells might be not enough to provide the necessary information to recover a polynomial solution of a desired order. Fortunately, the projection-based reconstruction is not the only way to obtain a polynomial solution of higher order from the underlying discontinuous Galerkin solutions. In a reconstructed DG method using a Taylor basis developed by Luo et al. [13–16] for the solution of the compressible Euler and Navier–Stokes equations on arbitrary grids, a higher order polynomial solution is reconstructed by use of a strong interpolation, requiring point values and derivatives to be interpolated on the face-neighboring cells. The resulting over-determined linear system of equations is then solved in the least-squares sense. This reconstruction scheme only involves von Neumann neighborhood, and thus is compact, simple, robust, and flexible. Like the projection-based reconstruction, the strong reconstruction scheme guarantees exact conservation, not only of the cell averages but also of their slopes due to a judicious choice of the Taylor basis. The latest hierarchical WENO-based rDG(P_nP_m) schemes [17,18] are designed not only to reduce the high computing costs associated with DG methods, but also to avoid spurious oscillations in the vicinity of strong discontinuities.

Indeed, DG methods are natural choices for solving hyperbolic systems, such as the compressible Euler equations. However, when it comes to elliptic or parabolic equations, such as the compressible Navier–Stokes equations, the DG formulation is far less certain and advantageous. Approaches made to resolve this issue could be found in the literature [5,14,19–28]. Those methods have introduced in some way the influence of the discontinuities in order to define correct and consistent diffusive fluxes. Unfortunately, all these methods seem to require substantially more computational effort than the classical continuous finite element methods, which are naturally more suited for the discretization of elliptic problems. There is also an approach where a scalar diffusion scheme is derived from a hyperbolic diffusion formulation [29,30]. It has been extended to higher-order in the context of the residual-distribution method [31], but has not been extended in the DG methods beyond second-order.

Over the last several years, an alternative approach to viscous discretizations, which reformulates the viscous terms as a first-order hyperbolic system (FOHS), was developed by Nishikawa [32–36], Nishikawa and Roe [37], Nakashima et al. [38], Liu and Nishikawa [39], Mazaheri and Nishikawa [40], Montecinos and Toro [41], Montecinos et al. [42], Toro and Montecinos [43], and Ahn et al. [44]. Note that the approaches in the references [41–43], present explicit ADER schemes for hyperbolic-diffusion systems with L_r, a free parameter defined as relaxation length, of $\mathcal{O}(h)$ rather than $\mathcal{O}(1)$. In their approach L_r (or T_r, another parameter as relaxation time) needs to depend on the mesh size in order to preserve the designed order of accuracy with explicit time stepping. Thus, their approach is different from the hyperbolic approach we present and discuss here. In the FOHS formulation, by including derivative quantities as additional variables, the equations are first formulated as a first order system (FOS). Then, it is rendered to be hyperbolic, which is the distinguished feature of the FOHS method from other FOS methods, by adding pseudo time derivatives to the first-order system. It thus generates a system of pseudo-time evolution equations for the solution and the derivatives in the partial differential equation (PDE) level, not in the discretization level as in DG methods. The hyperbolic reformulation in the PDE level would allow a dramatic simplification in the discretization as the well-established methods can be directly applied to the viscous terms. Moreover, the presented hyperbolic approach is not targeted at addressing stiff source terms already present in the original formulations. The approach introduces source terms which are not stiff for typical hyper-Re problems of small ν since $T_r = \mathcal{O}(1/\nu)$, in contrast to the hyperbolic approach used in Toro's work [43] where $T_r = \mathcal{O}(\nu)$. The FOHS method is especially attractive in the context of the DG methods since it allows the use of inviscid algorithms for the viscous terms and thus greatly simplifies the discretization of the compressible Navier–Stokes equations. Moreover, the FOHS method yields a numerical scheme that can achieve the same order of accuracy in the solution and its derivatives on irregular grids and high-quality noise-free gradients on such grids. This is a very important feature for unstructured-grid viscous simulations, where target quantities are derivatives, e.g., viscous stresses and heat fluxes.

A challenge in combining the DG method and the FOHS method lies in a very large number of discrete unknowns arising from both methods. For a scalar equation in two dimensions, the FOHS method introduces two derivatives as additional variables, and a DG(P_1) method introduces three degrees of freedom (DoFs) for each variable (solution, and two derivatives), resulting in the total of nine degrees of freedom. In 2015, the fourth author noticed that these degrees of freedom can be significantly reduced by unifying inter-related high-order moments of the derivative variables and extending the idea