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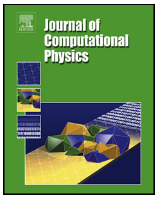

The frictional pebble game: An algorithm for rigidity percolation in saturated frictional assemblies

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A B S T R A C T

We present a fast, robust algorithm to determine rigid and floppy structures in two-dimensional (2D) assemblies of saturated frictional particles, based upon a generalisation (Lee and Streinu (2008) [11]) of the classical pebble game algorithm (Jacobs and Hendrickson (1997) [10]) for frictionless systems. By saturated, we refer to frictional systems for which all particle contacts involve sliding friction, which is relevant for certain frictional models and the study of incipient flow and stress-strain hysteresis. Whilst we do not present a mathematical proof of this method, we demonstrate validity via comparison with results from a classical method based on the network stiffness matrix over a diverse range of particle assemblies, including diluted bond networks and assemblies of cohesive, frictional particles.

1. Introduction

1.1. Scope

Since the seminal work of Maxwell [1], it is well known that the macroscopic properties of networks dominated by central forces is governed by the number of independent microscopic degrees of freedom in the system. This criterion has subsequently been applied to a wide array of physical systems, including structural frameworks [2], glass-like networks [3], frictionless particle assemblies [4], triangular lattices [5] and protein folding [6]. Counting these independent degrees of freedom is non-trivial due to the connectivity between constraints, leading to highly non-local behaviour. Hence percolation of rigid structures at the microscale demarcates the onset of macroscopic structural rigidity [3] or a jamming transition [4].

The classical approach to determining rigidity percolation revolves around analysis of the nullspace of the stiffness matrix $\mathcal{M}$ for elastic central forces (via e.g. diagonalization [7], singular value decomposition [2] or energy minimisation [8] methods). This approach aims to identify rigid and floppy structures in the assembly via an energy-free perturbation of particle position from equilibrium. A criterion for determining whether particle contacts are rigid or floppy is then imposed in terms of e.g. the deformation of the relative angle between particle contacts common to a given particle. However for large systems, due to numerical ill-posedness, a clear bandgap in the spectrum of these deformations does not occur, rendering identification of rigid/floppy structures imprecise, regardless of the accuracy of the method to determine the stiffness matrix nullspace. The non-local nature of rigidity percolation means that small errors are amplified at critical jamming transitions.

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1.2. Discrete approaches to rigidity percolation

In contrast, the constraint counting theorem of Laman [9] highlights the fundamentally discrete nature of this problem, hence the errors associated with continuous methods arise from numerical ill-posedness. This theorem treats the 2D particle network as a graph G, comprised of V vertices (particles) and E edges (contacts), and states [10]:

"*A graph G with* $2n - 3$ *edges is rigid in two dimensions if and only if no subgraph* G' *has more than* $2n' - 3$ *edges*".

Laman's theorem then forms the basis for independent constraint counting, and reflects the fact that the n particles in the network each contain two translational degrees of freedom (DOFs) as particle orientation is unimportant in frictionless networks, whilst there are also 3 trivial rigid body motions in 2D (associated with translation and rotation). Whilst such subgraph counting scales exponentially with the number of particles, Jacobs and Hendrickson [10] develop an $\mathcal{O}(n^2)$ pebble game algorithm for rapid identification of rigid clusters. This $(2, 3)$ pebble game has since been extended by Lee and Streinu [11] to general (k, l) pebble games for arbitrary constraint systems with an integer $k > 0$ DOFs per particle and $l \in [0, 2k)$ trivial motions of the global network.

Whilst the $(2, 3)$ pebble game solves rigidity percolation for 2D frictionless particle assemblies, extension to sliding frictional contacts is non-trivial due to a number of reasons. First, there is no known equivalent of Laman's theorem for frictional assemblies. Second, whilst sliding friction between particles involves tight coupling between translational and rotational degrees of freedom, macroscopic rigidity depends upon microscopic particle translation only. This subtlety demands counting of independent constraints with respect to coupled particle translation and reorientation, but rigidity must then be determined with respect to translational displacements only. Conversely, as shall be shown, the inclusion of rolling friction is straightforward due to the decoupling of rotational DOFs from translation DOFs for this friction mode.

1.3. Pebble games for frictional rigidity percolation

Motivated by the non-trivial role of particle orientation in frictional assemblies, Henkes et al. [12] propose that the $(3, 3)$ pebble game solves the rigidity percolation problem for arbitrary frictional structures, where frictional (frictionless) contacts correspond to double (single) edges between vertices. However, we find counterexamples such as the square array of frictional particles shown in Figure S2 in [12]. Here the $(3, 3)$ pebble game predicts the structure is floppy, whilst from the arguments above the particles are translationally rigid. We note that whilst such particle cycles and other open structures are rare in assemblies of repulsive or neutrally attractive particles such as granular matter, these occur frequently in cohesive networks such as strongly flocculated colloidal suspensions.

Hence, to our knowledge the analysis of rigidity percolation in frictional assemblies is an open problem. We also note that the presence of sliding friction renders relaxation methods significantly more ill-posed than for their frictionless counterparts, hence the need for integer-based methods is even more pressing. We propose an integer-based method for determining rigid structures in "saturated" frictional assemblies, where saturated refers to the fact that all particle contacts in the network involve sliding friction. We extend this method to the case where an arbitrary number of contacts between particles also involve rolling friction. The problem of saturated sliding friction is relevant to (i) ratchet-shaped sliding friction models [13] in which the friction increment is always finite, or (ii) deformation initialisation and hysteresis problems, where frictional interactions are all in the elastic regime. Such analysis is integral to the macroscopic study of the hysteresis of frictional materials. Saturated frictional and frictionless rigidity analyses also form bounds for frictional systems with an arbitrary number of contacts with sliding friction, and these two extrema reflect material behaviour under initial and large deformations respectively. In this way, these two analysis allow the extremum of the deformation of frictional materials to be probed; for large, monotonic, deformations all sliding contacts eventually become frictionless, whereas under immediate deformation reversal, and contacts are fully frictional. Lastly, we expect that solution of the pebble game algorithm for saturated frictional assemblies provides some insights for solution of the more general arbitrary problem involving partially saturated frictional assemblies.

2. Rigidity percolation in frictional particle assemblies

2.1. Frictional particle mechanics and constraints

To test and develop a frictional pebble game algorithm we consider frictional rigidity percolation in two different systems: (i) the classical bond diluted triangular network shown in Fig. 1(a), and (ii) a network of cohesive frictional particles shown in Fig. 1(b)–(d). For both of these systems we consider a specific mechanistic model for the normal and frictional forces between particles based upon a model for the consolidation of concentred colloidal suspensions [13]. Whilst this model is specific to a particular physical system, in Section 3 we show that the resultant dynamical stiffness matrix (and specifically the associated null modes) which governs network rigidity applies to generic frictional particle assemblies. Hence testing of the frictional pebble game algorithm herein is relevant to a broad set of frictional systems.

Both systems considered herein are comprised of 2D arrays of mono-disperse discs (particles) which involve short-ranged pairwise frictional and normal forces between contacting particles. Upon contact, a bond is immediately initiated between a pair of particles (labelled (i, j)) which has zero force and moment exactly at the instance of contact. As illustrated in Fig. 2, any translational and/or rotational displacement of either particle after contact can then result in a non-zero normal $\mathbf{F}_N^{(i,j)}$ or