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Higher-order finite volume differential operators with selective upwinding on the icosahedral spherical grid

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ABSTRACT

The icosahedral hexagonal grid is a quasi-uniform discretization of the sphere, generated as the dual of a triangular grid derived by successively refining the faces of an icosahedron embedded in the sphere. This grid contains twelve pentagonal cells and a large number of hexagonal cells, and the bounded area ratio between the smallest and largest cells eliminates polar singularities.

This structure, however, makes the derivation of high-order numerical methods more difficult than for grids with a more regular structure. This work progresses towards the goal of high-order operators for the dynamical core of a global atmospheric model by developing a nonstaggered finite volume method on this grid, featuring upwinding applied only when necessary for stability, using an edge-centered reconstruction that directly accommodates the grid's intrinsic curvature.

These operators compute the cell-average divergence, curl, and gradient with accuracy between second and fourth order for test functions. When applied to shallow water test cases, the resulting method is between third and fourth order accurate for the transport equation and between second and third order accurate for the full shallow water system.

1. Introduction

The dynamical cores of global atmospheric forecast models are moving away from their traditional global griddings, driven by advances in supercomputing technology and the need for grids that support both high resolution and distributed memory (cluster) computing technologies. Latitude–longitude grids are bottlenecked by nonuniform resolution near the poles, whereas global spectral methods are limited by the computational complexity of the spherical harmonic transforms.

The quasi-uniform grid based on the global icosahedral hexagonal discretization [1] shows promise as a candidate for highly-scalable dynamical cores. This grid is constructed by successive refinements of the icosahedron, with nodes re-projected to the surface of the sphere. The result is a semi-structured but nonrectangular grid where each vertex is connected to either five or six neighbours, and the distance ratio between the largest and smallest edge-lengths remains bounded. Conventionally, the dual grid consisting of node centered polygons forms the set of computational cells. Further optimization is possible via adjustment of grid nodes (preserving connectivity) to improve numerical properties [2].

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Shallow water equations

The accurate solution of the shallow water equations is the first step towards creating a dynamical core suitable for the three-dimensional atmosphere. The shallow water system is a computationally tractable two-dimensional system of equations, but it captures the dynamics of gravity, Kelvin, and Rossby waves – the dominant features of horizontal atmospheric motion. In this system, the full three-dimensional Navier–Stokes equations are simplified [3] by assuming that the fluid lies within a thin layer, such that the length scale of horizontal motion is much greater than fluid depth allowing the neglect of vertical motion.

Written in the flux form for fluid depth and vorticity/divergence form for fluid velocity, the shallow water system of equations is [4]:

$$\frac{\partial h}{\partial t} = -\nabla \cdot (h\vec{u}) \tag{1a}$$

and

$$\frac{\partial \vec{u}}{\partial t} = -\big(f + \hat{k} \cdot (\nabla \times \vec{u})\big)(\hat{k} \times \vec{u}) - \nabla\big(g(h+H) + \tfrac{1}{2}\vec{u} \cdot \vec{u}\big), \tag{1b}$$

where $h(\vec{x})$ is the local fluid depth with $H(\vec{x})$ the height of any underlying topography, $\vec{u}(\vec{x})$ is the local fluid velocity, f is the Coriolis parameter, $\hat{k}$ is the local vertical unit vector, and g is the gravitational acceleration towards the center of the sphere.

Existing work

A number of authors have previously described finite volume schemes on the icosahedral grid, each exhibiting different desirable properties. Among these models, the TRiSK finite volume scheme [5–8] is a staggered finite-volume method built to preserve so-called mimetic properties, including preserving steady geostrophic modes and advecting potential vorticity consistently with mass. Related work by Ringler et al. [9] adapted this discretization to conserve energy, and Aechtner et al. [10] incorporated local grid refinement.

The MPDATA finite-volume scheme [11] is a non-staggered method that supports a fully unstructured mesh on the sphere, and it achieves second-order accuracy by deriving a nonlinear antidiffusive flux that corrects the leading-order errors in the first-order upwind flux. This method was applied to the shallow-water equations by Szmelter and Smolarkiewicz [12] and the full three-dimensional Euler system by Smolarkiewicz et al. [13].

Achieving better than second-order accuracy is difficult on this grid, and few authors present shallow-water methods with better than second-order accuracy. The multi-moment finite volume method [14–16] achieves third and potentially fourth-order accuracy for a non-staggered finite volume method by adding collocation points along the edge of each element. The shallow-water equations are solved there using intra-element polynomial fits to evaluate derivatives, while a single degree of freedom at the barycenter of each element is dedicated to preserving the evolution of cell average quantities.

For the transport equation, there has been more development towards higher-order solutions. Skamarock and Gassmann [17] developed third and fourth-order operators for unstructured spherical grids based on the Cartesian formulation of WRF. Additionally, Miura [18] and Miura and Skamarock [19] have described a third-order, upwind biased method for transport on the spherical icosahedral grid, and very recently Waruszewski et al. [20] has developed a third-order formulation of MPDATA for the transport problem. For this problem, higher-order methods are realized through higher-order flux operators. This approach has been implemented in solvers of the full shallow water system [21–23]. While it tends to improve the models' respective errors, it does not improve the overall convergence rate beyond second-order.

Objective & organization

High-order fluxes are a necessary but not sufficient condition for a finite-volume method to solve the shallow water equations to a comparable order of accuracy: assumptions made in the derivation of volume elements or in the specification of the differential operators ultimately limits convergence to something less than the accuracy of the fluxes. After briefly describing the grid generation procedure, this work develops discrete operators in section 2.1 for the two-dimensional surface gradient (∇f for scalar function f), divergence ($\nabla \cdot \vec{u}$, for a vector field $\vec{u}$), and curl ($\hat{k} \cdot \nabla \times \vec{u}$, where $\hat{k}$ is again the unit vertical vector) that converge to third and fourth-order when applied to test functions and to at least second order when applied to the shallow water equations for standard test problems.

Similarly to the models of Tomita et al. [24] and Pudykiewicz [25], the numerical scheme described here is based on a non-staggered discretization where each discrete degree of freedom represents a cell-averaged quantity. This framework is the natural one for the use of Stokes theorem to define the cell-average operators, and that derivation is the subject of section 2.2.