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A first-order asymptotic preserving scheme for front propagation in a one-dimensional kinetic reaction–transport equation

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ABSTRACT

In this work, we propose an asymptotic preserving scheme for a nonlinear kinetic reaction–transport equation, in the regime of sharp interface. With a nonlinear reaction term of KPP-type, a phenomenon of front propagation was proven in [9]. This behaviour can be highlighted by considering a suitable hyperbolic limit of the kinetic equation, using a Hopf–Cole transform. It was proven in [6,8,11] that the logarithm of the distribution function then converges to the viscosity solution of a constrained Hamilton–Jacobi equation.
The hyperbolic scaling and the Hopf–Cole transform make the kinetic equation stiff. Thus, the numerical resolution of the problem is challenging, since the standard numerical methods usually lead to high computational costs in these regimes. *Asymptotic Preserving* (AP) schemes have typically been introduced to deal with this difficulty, since they are designed to be stable along the transition to the macroscopic regime. The scheme we propose is adapted to the non-linearity of the problem, enjoys a discrete maximum principle, and solves the limit equation in the sense of viscosity. It is based on a dedicated micro–macro decomposition attached to the Hopf–Cole transform. As it is well adapted to the singular limit, our scheme is able to cope with singular behaviours in space (sharp interface), and possibly in velocity (concentration in the velocity distribution). Various numerical tests are proposed to illustrate the properties and the efficiency of our scheme.

1. Introduction

We are interested in designing a numerical scheme for a nonlinear kinetic equation in the asymptotic regime. The model we consider is a nonlinear transport–reaction equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \rho(t,x)M(v) - f(t,x,v) + r\rho(t,x)(M(v) - f(t,x,v)), \tag{1}$$

with $r \geq 0$, supplemented with initial data $f(0,x,v) = f_{\text{in}}(x,v) = \rho_{\text{in}}(x)M(v)$. Such models have been introduced in [34,20,15]. The asymptotic regime of (1) was studied in [8,6,11], both in the linear case $r = 0$, and in the nonlinear case $r > 0$. In (1), the distribution function f, which depends on $t > 0$, $x \in \mathbb{R}^d$, and $v \in V$, where V is a bounded symmetric set of $\mathbb{R}^d$, represents the density of particles at time t, at the position x, and with velocity v. The macroscopic density of particles is defined by

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$$\rho(t,x) = \langle f \rangle := \int_{v\in V} f(t,x,v)\mathrm{d}v, \quad t \geq 0, \ x \in \mathbb{R}^d.$$

Note that the brackets $\langle\cdot\rangle$ denote integration in velocity throughout the paper. For $r = 0$, equation (1) describes the evolution of the density of particles moving according to a velocity-jump process. Indeed, the motion of a particle is composed of phases of free transport, of *run* phases (with a velocity v) and of *tumble* phases (in which the particle changes velocity instantaneously). The post-tumbling velocity is chosen randomly, according to a given probability density M. We assume that M is even, non-negative, and continuous. Moreover, it satisfies

$$\langle M \rangle = 1, \quad \langle vM \rangle = 0, \tag{2}$$

and we will suppose that

$$\inf_{v\in V} M(v) > 0. \tag{3}$$

Condition (3) is a technical condition, required to stay within the context of the asymptotic analysis of (1), as in [6,8], which is the aim of this paper. When (3) is not satisfied, the asymptotic equation is modified, see [11]. Note that some comments and numerical tests on the case $\inf_{v\in V} M(v) = 0$ are proposed in Section 6.4.

Equation (1) is complemented with a reaction term in the case $r > 0$. It takes into account creation of new particles at rate r, and local quadratic saturation. Initial velocity of new particles is drawn randomly from M. Averaging with respect to velocity leads to the classical logistic growth $r\rho(1-\rho)$.

We consider the kinetic equation (1) under a hyperbolic scaling $(t,x,v) \longmapsto (t/\varepsilon, x/\varepsilon, v)$. Indeed, since we are interested in the study of propagation phenomena in (1), the time and space scale have to be equal. The kinetic equation (1) then reads

$$\partial_t f^\varepsilon(t,x,v) + v\cdot\nabla_x f^\varepsilon(t,x,v) = \frac{1}{\varepsilon}\left(\rho^\varepsilon(t,x)M(v) - f^\varepsilon(t,x,v) + r\rho^\varepsilon(t,x,v)(M(v) - f^\varepsilon(t,x,v))\right). \tag{4}$$

The propagation of fronts for (1) was studied in [9]. To study the asymptotic behaviour of (4) when ε goes to 0, an analogy is made in [8,6] with the sharp front limit of the Fisher–KPP equation. A *WKB ansatz* is introduced, leading to the so-called approximation of geometric optics (see [17,19]). It consists of rewriting the distribution function f^ε as

$$f^\varepsilon = M\mathrm{e}^{-\psi^\varepsilon/\varepsilon}. \tag{5}$$

The equation satisfied by ψ^ε in the limit $\varepsilon \to 0$ is then studied. In the case of the kinetic equation (4), if

$$0 \leq f^\varepsilon(0,\cdot,\cdot) \leq M,$$

a maximum principle ensures that ψ^ε is well defined and remains non-negative for all $t \geq 0$, see [6]:

Proposition 1. *Let $r \geq 0$ and let $\psi_{\mathrm{in}} \in \mathrm{Lip}(\mathbb{R}^d \times V)$, the Hopf–Cole transform (5) of f_{in}, bounded. Let $f^\varepsilon = M\mathrm{e}^{-\psi^\varepsilon/\varepsilon}$ a solution of (4). Then the phase ψ^ε is uniformly locally Lipschitz, and the following* a priori *bound holds*

$$\forall t \geq 0, \quad 0 \leq \psi^\varepsilon(t,\cdot,\cdot) \leq \|\psi_{\mathrm{in}}\|_\infty. \tag{6}$$

In the case of the Fisher–KPP equation, it has been proven that the function ψ^ε converges to a limit function ψ^0, which is the viscosity solution of a Hamilton–Jacobi equation, see [17,2,3,35,13]. Moreover, in the asymptotic regime, the settled population $\rho \sim 1$ is contained in the nullspace of ψ^0, see [16,4,18].

The analysis of propagation phenomena at the mesoscopic scale is motivated by concentration waves of chemotactic bacteria, as observed experimentally in [33]. Here, the model under investigation does not contain any chemotactic effect, but takes into account cell division. It satisfies the maximum principle, hence it is more amenable for mathematical analysis, following the seminal works by Kolmogorov, Petrovsky, Piskunov [26], and Aronson, Weinberger [1]. The first analytical works, where travelling waves are constructed, are [34] and [15]. Note that the latter develops a micro–macro decomposition to handle the construction of travelling waves near the diffusive regime. We also refer to [20], and references therein, for a more general presentation of reaction transport equations in biology.

The asymptotic behaviour of (4) in the limit $\varepsilon \to 0$ was established rigorously in [8,6]. Before stating the main theorem, let us highlight that the formation of fronts if $r > 0$ can be understood with very formal considerations on (4). Indeed, when ε goes to 0, supposing that the distribution function f^ε and its density ρ^ε converge respectively to f^0 and ρ^0, we have formally at order 0 in ε

$$\rho^0 M - f^0 + r\rho^0(M - f^0) = 0, \tag{7}$$