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A posteriori error estimation for an augmented mixed-primal method applied to sedimentation–consolidation systems ☆

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ABSTRACT

In this paper we develop the *a posteriori* error analysis of an augmented mixed-primal finite element method for the 2D and 3D versions of a stationary flow and transport coupled system, typically encountered in sedimentation–consolidation processes. The governing equations consist in the Brinkman problem with concentration-dependent viscosity, written in terms of Cauchy pseudo-stresses and bulk velocity of the mixture; coupled with a nonlinear advection – nonlinear diffusion equation describing the transport of the solids volume fraction. We derive two efficient and reliable residual-based *a posteriori* error estimators for a finite element scheme using Raviart–Thomas spaces of order k for the stress approximation, and continuous piecewise polynomials of degree $\leq k+1$ for both velocity and concentration. For the first estimator we make use of suitable ellipticity and inf–sup conditions together with a Helmholtz decomposition and the local approximation properties of the Clément interpolant and Raviart–Thomas operator to show its reliability, whereas the efficiency follows from inverse inequalities and localisation arguments based on triangle-bubble and edge-bubble functions. Next, we analyse an alternative error estimator, whose reliability can be proved without resorting to Helmholtz decompositions. Finally, we provide some numerical results confirming the reliability and efficiency of the estimators and illustrating the good performance of the associated adaptive algorithm for the augmented mixed-primal finite element method.

1. Introduction

The phenomenon of gravitational sedimentation of relatively small particles within viscous fluids is of considerable importance in a number of diverse applications related for instance to wastewater treatment, mineral processing, volcanology, or hemodynamics. In this process the suspended mixture is separated into the solid particles going to the bottom of the vessel and the viscous fluid remaining on the top. Using the formalism of mixtures, one can assume that both fluid and solid phases are superimposed continua, and regarding the problem from a macroscopic viewpoint, the governing equations involving momentum and mass conservation of the phases can be conveniently recast in the form of one momentum

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and one mass equation for the mixture, together with a mass conservation equation for the solids concentration (see e.g. [10,13,35]). As the flow regime under consideration is viscous, laminar, and in the presence of a background porosity in the vessel, the PDE system under consideration consists of Brinkman equations with variable viscosity coupled with a nonlinear advection – nonlinear diffusion equation describing the transport of the volumetric fraction of the solids.

We have recently analysed in [2], the solvability of a strongly coupled flow and transport system encountered in such continuum-based models for sedimentation–consolidation processes. There we have considered the steady-state regime of the process and we have proposed an augmented variational formulation where the main unknowns given by the Cauchy pseudo-stress and bulk velocity of the mixture, and the solids volume fraction, which are sought in $\mathbb{H}(\mathbf{div};\Omega)$, $\mathbf{H}^1(\Omega)$, and $\mathrm{H}^1(\Omega)$, respectively. Fixed point arguments, certain regularity assumptions, and some classical results concerning variational problems and Sobolev spaces are combined to establish the solvability of the continuous and discrete coupled formulations. Consequently, the rows of the Cauchy stress tensor were approximated with Raviart–Thomas elements of order k, whereas the velocity and solids concentration were discretised with continuous piecewise polynomials of degree $\le k+1$. Suitable Strang-type estimates are employed to derive optimal *a priori* error estimates for the solution of the Galerkin scheme.

The purpose of this work is to provide reliable and efficient residual-based *a posteriori* error estimators for the steady sedimentation–consolidation system studied in [2]. Estimators of this kind are frequently employed to guide adaptive mesh refinement in order to guarantee an adequate convergence behaviour of the Galerkin approximations, even under the eventual presence of singularities. The global estimator $\boldsymbol{\eta}$ depends on local estimators η_T defined on each element T of a given mesh $\mathcal{T}_h$. Then, $\boldsymbol{\eta}$ is said to be efficient (resp. reliable) if there exists a constant $C_{\tt eff}>0$ (resp. $C_{\tt rel}>0$), independent of meshsizes, such that

$$C_{\tt eff}\,\boldsymbol{\eta} + \text{h.o.t.} \le \|error\| \le C_{\tt rel}\,\boldsymbol{\eta} + \text{h.o.t.}\,,$$

where h.o.t. is a generic expression denoting one or several terms of higher order. Up to the authors knowledge, a number of *a posteriori* error estimators specifically targeted for non-viscous flow equations (e.g., Darcy) coupled with transport problems, are available in the recent literature [9,20,31,38,42]. However, only [11,32] and [3] are devoted to the *a posteriori* error analysis for coupled viscous flow-transport problems. In particular, we derive in [3], two efficient and reliable residual-based *a posteriori* error estimators for an augmented mixed-primal finite element approximation of a stationary viscous flow and transport problem, which serves as a prototype model for sedimentation–consolidation processes and other phenomena where the transport of species concentration within a viscous fluid is of interest.

In this paper, as well as in [3,4], we make use of ellipticity and inf–sup conditions together with a Helmholtz decomposition, local approximation properties of the Clément interpolant and Raviart–Thomas operator, and known estimates from [8], [22], [26], [28] and [29], to prove the reliability of a residual-based estimator. Then, inverse inequalities, the localisation technique based on triangle-bubble and edge-bubble functions imply the efficiency of the estimator. Alternatively, we deduce a second reliable and efficient residual-based *a posteriori* error estimator, where the Helmholtz decomposition is not employed in the corresponding proof of reliability. The rest of this paper is organised as follows. In Section 2, we first recall from [2] the model problem and a corresponding augmented mixed-primal formulation as well as the associated Galerkin scheme. In Section 3, we derive a reliable and efficient residual-based *a posteriori* error estimator for our Galerkin scheme. A second estimator is introduced and analysed in Section 4. Next, the analysis and results from Section 3 and 4 are extended to the three-dimensional case in Section 5. Finally, in Section 6, our theoretical results are illustrated via some numerical examples, highlighting also the good performance of the scheme and properties of the proposed error indicators.

2. The sedimentation–consolidation system

Let us denote by $\Omega \subseteq \mathrm{R}^n$, $n=2,3$ a given bounded domain with polyhedral boundary $\Gamma = \bar{\Gamma}_{\mathrm{D}} \cup \bar{\Gamma}_{\mathrm{N}}$, with $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$ and $|\Gamma_{\mathrm{D}}|, |\Gamma_{\mathrm{N}}| > 0$, and denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. In particular, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. By $\mathbf{M}, \mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M. We recall that the space

$$\mathbb{H}(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega): \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\}\,,$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}$$

is a Hilbert space. As usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes both the Euclidean norm in R^n and the Frobenius norm in $\mathrm{R}^{n\times n}$.

2.1. The governing equations

The following model describes the steady state of the sedimentation–consolidation process consisting on the transport and suspension of a solid phase into an immiscible fluid contained in a vessel Ω (*cf.* [2]). The flow patterns are influenced by