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Mesh-free semi-Lagrangian methods for transport on a sphere using radial basis functions

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ABSTRACT

We present three new semi-Lagrangian methods based on radial basis function (RBF) interpolation for numerically simulating transport on a sphere. The methods are mesh-free and are formulated entirely in Cartesian coordinates, thus avoiding any irregular clustering of nodes at artificial boundaries on the sphere and naturally bypassing any apparent artificial singularities associated with surface-based coordinate systems. For problems involving tracer transport in a given velocity field, the semi-Lagrangian framework allows these new methods to avoid the use of any stabilization terms (such as hyperviscosity) during time-integration, thus reducing the number of parameters that have to be tuned. The three new methods are based on interpolation using 1) global RBFs, 2) local RBF stencils, and 3) RBF partition of unity. For the latter two of these methods, we find that it is crucial to include some low degree spherical harmonics in the interpolants. Standard test cases consisting of solid body rotation and deformational flow are used to compare and contrast the methods in terms of their accuracy, efficiency, conservation properties, and dissipation/dispersion errors. For global RBFs, spectral spatial convergence is observed for smooth solutions on quasi-uniform nodes, while high-order accuracy is observed for the local RBF stencil and partition of unity approaches.

1. Introduction

Radial basis function (RBFs) methods have been used for over a decade to solve partial differential equations (PDEs) on spheres. These methods can broadly be classified into *global* RBF collocation methods [1–5], RBF-generated finite difference (RBF-FD) methods [6–10], and more recently RBF-partition of unity (RBF-PU) collocation methods [11]. Global RBF methods when used with infinitely-smooth RBFs show spectral convergence on smooth problems at the cost of *dense* differentiation matrices; in contrast, RBF-FD and RBF-PU methods produce sparse differentiation matrices and high-order algebraic convergence rates. All of these methods can use "scattered" nodes in their discretizations of a sphere, and have the benefit of being independent of any surface-based coordinate system. They thus avoid any unnatural grid clustering and do not suffer from any coordinate singularities.

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In this paper, we present three new RBF methods with similar benefits for numerically solving the transport equation on the surface of a sphere in an incompressible velocity field. For the unit sphere $\mathbb{S}^2$, this PDE is given

$$\frac{Dq}{Dt} = 0, \quad \frac{D}{Dt} := \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla_{\mathbb{S}^2}, \tag{1}$$

where q is the scalar quantity being transported, $\boldsymbol{u}$ is a surface divergence-free vector field that is tangent to $\mathbb{S}^2$, and $\nabla_{\mathbb{S}^2}$ denotes the surface gradient operator on $\mathbb{S}^2$. Since global atmospheric flows are dominated by the horizontal advection process, the numerical solution to the transport problem is a fundamental part of any solver for these flows.

Currently, all RBF discretizations for the transport equation (and more general hyperbolic equations like the shallow water equations) on a sphere suffer from the same drawback: the eigenvalues of the differentiation matrices corresponding to the surface gradient operator may in general have positive real parts, leading to either a slow or rapid onset of instability during a numerical simulation [1]. As of this paper, the only known approach to rectify this problem is to add an artificial hyperviscosity of the form $(-1)^{(k+1)}\gamma\Delta^k$ in the right hand side of the PDE, where $k \geq 2$ is an integer and $\gamma > 0$ is some small real number that scales *inversely* with the total number of nodes N. The intuition here is that higher powers of the Laplacian Δ will damp out the eigenvectors associated with the rogue eigenvalues of the discretized surface gradient, while leaving the others essentially untouched [6,7,12,13]. With global RBFs, the hyperviscosity operator typically takes the form of γA^{-1}, where A is the global RBF interpolation matrix whose inverse mimics the properties of high powers of the Laplacian [6]; a similar approach can be employed for the RBF-PU method [11]. Unfortunately, for a given PDE and node set on a sphere, the precise values of γ and k required to stabilize the numerical solution may need to be determined by trial and error, which can add to the computational expense of the method.

A common way to naturally stabilize local Eulerian methods for transport is to use "upwinding", which uses dynamic direction-dependent information about the flow field. However, this form of upwinding typically requires an underlying mesh and so is impractical for truly mesh-free local methods like RBF-FD and RBF-PU collocation. An important class of methods that naturally possess upwinding are semi-Lagrangian (SL) schemes, widely acknowledged for their formal independence from the CFL stability condition [14]. SL methods have successfully been used for simulating various problems in fluid dynamics, especially in numerical weather prediction and climate modeling, from simple tracer transport to more complex problems involving wide ranges of spatiotemporal scales and intricate forcing terms, e.g. [14–18]. For problems on a sphere, SL methods have generally used latitude–longitude grids and spherical coordinate systems (e.g. [19–21]), or other regular surface-based grids and local surface-based coordinate systems (e.g. [22–24]), which can lead to a loss of accuracy because of singularities that arise in the mappings from the physical sphere to the surface-based coordinate systems.

In this paper, we present three new high-order SL methods for transport on a sphere based on interpolation with global RBFs, local RBFs, and RBF-PU methods. These methods are mesh-free, allowing for scattered node discretizations, and are formulated entirely in Cartesian coordinates so as to avoid any surface based coordinate singularities. The local RBF and RBF-PU methods also allow for a type of "p-refinement" for increasing the accuracy for a given fixed set of discretization nodes. We demonstrate that the SL framework lends our new methods both accuracy and intrinsic stability, thereby eliminating the need for a hyperviscosity term. For the local RBF and RBF-PU methods, we propose using "scale-free" RBFs appended with spherical harmonics (an idea related to that of Flyer et al. [25] for planar domains) to further reduce the number of tuning parameters (*i.e.*, the shape-parameter) and to bypass so-called error stagnation. We compare and contrast all three methods using three standard test cases from the literature—solid-body rotation of a cosine bell from [26] and deformational flow of two bells from [27]. The focus of these comparisons is on the overall accuracy, dissipation and dispersion properties, mass conservation, and computational cost. We find that the computational costs of these new methods are comparable to those of existing RBF collocation and finite-difference techniques. In particular, we find that the local RBF and RBF-PU methods are highly scalable to large node sets. We note that RBFs have previously been used for SL advection in [28], but the focus there was on planar domains and global RBF methods. Additionally, they have been used in a conservative SL advection method in [29] for planar domains using Voronoi cells and local thin plate splines. This is the first application of RBFs to SL transport on a sphere with accurate and scalable numerical methods and in a entirely mesh-free formulation.

We note that, while SL methods are applicable to a wide class of advection dominated problems, they have some limitations, the primary one being a lack of local conservation unless explicitly formulated in a conservative manner. However, the large-time-step conservative SL methods are computationally expensive [30].

The paper is organized as follows. In the next section, we review the global RBF, local RBF and RBF-PU methods in the context of interpolation. In Section 3, we review the SL advection technique and discuss how to use the three RBF methods within the SL framework in an efficient fashion. In Section 4, we compare and contrast the new SL methods on three standard test cases for transport on the unit sphere. Finally, we conclude with a summary of our results and future research directions in Section 5.

2. Global, local, and partition of unity RBF interpolation on $\mathbb{S}^2$

RBFs are a well-established method for interpolating/approximating data over a set of "scattered" nodes $X = \{\boldsymbol{x}_j\}_{j=1}^N \subset \Omega \subseteq \mathbb{R}^d$. The standard method uses linear combinations of shifts of a kernel $\phi : \Omega \times \Omega \to \mathbb{R}$ with the property that $\phi(\boldsymbol{x}, \boldsymbol{y}) := \phi(\|\boldsymbol{x} - \boldsymbol{y}\|)$ for $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. Kernels with this property are referred to as *radial kernels*