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Arbitrary high order accurate space–time discontinuous Galerkin finite element schemes on staggered unstructured meshes for linear elasticity

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ABSTRACT

In this paper we propose a new high order accurate space–time discontinuous Galerkin (DG) finite element scheme for the solution of the linear elastic wave equations in first order velocity-stress formulation in two and three-space dimensions on *staggered* unstructured triangular and tetrahedral meshes. The method reaches arbitrary high order of accuracy in both space and time via the use of space–time basis and test functions. Within the staggered mesh formulation, we define the discrete velocity field in the control volumes of a primary mesh, while the discrete stress tensor is defined on a face-based staggered dual mesh. The space–time DG formulation leads to an *implicit* scheme that requires the solution of a linear system for the unknown degrees of freedom at the new time level. The number of unknowns is reduced at the aid of the Schur complement, so that in the end only a linear system for the degrees of freedom of the velocity field needs to be solved, rather than a system that involves both stress and velocity. Thanks to the use of a spatially staggered mesh, the stencil of the final velocity system involves only the element and its direct neighbors and the linear system can be efficiently solved via matrix-free iterative methods. Despite the necessity to solve a linear system, the numerical scheme is still computationally efficient. The chosen discretization and the linear nature of the governing PDE system lead to an unconditionally stable scheme, which allows large time steps even for low quality meshes that contain so-called sliver elements. The fully discrete staggered space–time DG method is proven to be *energy stable* for any order of accuracy, for any mesh and for any time step size. For the particular case of a simple Crank–Nicolson time discretization and homogeneous material, the final velocity system can be proven to be symmetric and positive definite and in this case the scheme is also exactly *energy preserving*. The new scheme is applied to several test problems in two and three space dimensions, providing also a comparison with high order explicit ADER-DG schemes.

1. Introduction

Even nowadays the accurate simulation of elastic wave propagation in heterogeneous media involving complex geometries is a very challenging task. In the past several numerical methods have been developed in order to solve the linear elasticity equations. Some classical finite difference methods can be found in [1–3] and for further extensions and develop-

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ments see, e.g. [4–10]. Concerning the class of pseudo-spectral methods we refer the reader to [11,12]. The spectral finite element method, originally introduced by Patera in [13], was applied to linear elastic wave propagation in a well-known series of papers, see e.g. [14–18] and references therein. We also mention here alternative developments in the field of stabilized continuous finite elements for elastic and acoustic wave propagation based on the velocity stress formulation, see e.g. [19–21].

A major challenge in the numerical simulation of linear elastic waves is the ability of the numerical scheme to accurately propagate complex wave patterns over long distances and for very long times. Therefore, the use of high order schemes in both space and time is necessary. For a quantitative accuracy analysis of high order schemes applied to elastic wave propagation, see e.g. [22,23]. The analysis is based on the misfit criteria developed in [24,25]. For an alternative study of high order DG schemes applied to wave propagation problems we refer to [26].

Another challenge is the discretization of complex geometries including both, complex surface topography as well as complex sub-surface fault structures. In this case, the use of unstructured simplex meshes composed of triangles or tetrahedra seems to be beneficial concerning the problem of mesh generation in complex geometries. Concerning high order explicit discontinuous Galerkin (DG) finite element schemes for linear elastic wave propagation on general unstructured meshes the reader is referred to [27–31] and to [32–34]. However, since the previous methods are *explicit*, they are only stable under a CFL-type stability condition on the time step that depends on the mesh quality as well as the polynomial approximation degree used. In particular, unstructured simplex meshes for complex 3D geometries may contain so-called sliver elements, which are tiny elements with very bad aspect ratio and which look like needles or thin plates. In the case of explicit time discretizations, such elements can be efficiently treated only at the aid of time-accurate local time stepping (LTS), see e.g. [31,35–39]. In this paper, we try to solve this problem in a different way using an efficient high order accurate *implicit* time discretization.

Our work is inspired by a new class of high order accurate semi-implicit discontinuous Galerkin finite element schemes on staggered meshes recently introduced in [40–46] for the numerical solution of the shallow water equations, the incompressible and the compressible Navier–Stokes equations. Being semi-implicit, the previous methods allow large time steps. Furthermore, the use of an edge-based staggered grid allows to connect the discrete divergence operator with the discrete gradient operator. This leads to some interesting properties of the final pressure system that needs to be solved, which becomes symmetric and positive definite. The use of staggered meshes is state of the art for many finite difference schemes used in computational fluid dynamics [47–57] as well as for seismic wave propagation [6,58–60]. However, at present staggered meshes are still almost unknown in the context of high order discontinuous Galerkin finite element methods for wave propagation. Apart from the above-mentioned references on semi-implicit staggered DG schemes [40–46], the authors are only aware of [61–65] and references therein concerning high order DG schemes for wave propagation using edge-based staggered grids. For central DG schemes, which use a vertex-based grid staggering, the reader is referred to [66,67]. However, none of those references uses space–time discontinuous Galerkin finite elements, where the basis and test functions depend not only on space, but both on space and time. The concept of space–time DG schemes was introduced by van der Vegt et al. for computational fluid dynamics in [68–72] and has been subsequently analyzed e.g. in [73,74]. The first application of space–time DG schemes to elastodynamics on collocated grids has been reported in [75,76], but to the best of our knowledge there exists no space–time DG scheme for the linear elastic wave equations on staggered grids so far. It is the aim of this paper to design and analyze the properties of such methods.

More precisely, in this paper we extend the idea of staggered semi-implicit space–time discontinuous Galerkin methods for the Navier–Stokes equations [42,43,45,44] to linear elasticity. While the velocity field is discretized on the main grid, the stress tensor is defined on a face-based staggered *dual* mesh. The governing PDE system is linear and all terms are taken implicitly. Inserting the discrete evolution equations for the stress tensor into the discrete momentum equation leads to one single linear system for the velocity field via the application of the Schur complement. Once the velocity field at the new time is known, one can readily update the stress tensor using an explicit formula. The good properties of the main system already observed in [43,45] are achieved also in this case. The resulting numerical scheme is shown to be *energy stable* for any polynomial degree in space and time. A remarkable particular case can be obtained by using arbitrary high order polynomials in space combined with a second order Crank–Nicolson time discretization. For this special case the method becomes exactly energy preserving and the main system becomes symmetric and positive definite. We also present a simple and efficient physics-based preconditioner that is useful in the presence of sliver elements.

The rest of this paper is organized as follows: in Section 2 we present the governing PDE system and in Section 3 we introduce the staggered grid that is used in our approach, as well as the chosen basis functions. In Section 4 we present the numerical scheme and analyze its properties in Section 5. In Section 6 we show numerical results for several test problems in two and three space dimensions. We compare all numerical results obtained with our new high order staggered space–time DG scheme with those obtained by a high order explicit ADER-DG scheme on unstructured meshes. The paper closes with some concluding remarks and an outlook to future work in Section 7.

2. Governing equations

Based on the theory of linear elasticity, see e.g. [77], the governing partial differential equations for the wave propagation in a linear elastic medium without attenuation can be written in compact first order velocity-stress formulation based on the Hooke law and the momentum conservation law. They read