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An efficient boundary element formulation for doubly-periodic two-dimensional Stokes flow with pressure boundary conditions

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ABSTRACT

We present an efficient formulation of the boundary element method (BEM) for calculation of two-dimensional pressure-driven Stokes flow in a doubly-periodic domain. In contrast to similar methods which require *a priori* knowledge of the mean fluid velocity, this formulation is based on knowledge of the mean pressure gradient only. We present a method of calculating the permeability tensor without the need to specify either mean velocity or pressure gradient. We discuss optimality of the splitting parameter in the doubly-periodic Green's function, with regard to the numerical overhead required for the BEM, and in most cases find a 3–10 fold increase in computational efficiency compared to the splitting parameter used in standard BEM formulations.

1. Introduction

The Stokes equations provide a well-established approximation for flows with small Reynolds number, $\mathrm{Re} \ll 1$. Stokes flow problems in spatially periodic domains are important to many applications [16,14,12]. Here, the fluid is driven through a periodic array of solid inclusions by a gradient in pressure (or gravity) pointing in a direction of periodicity.

The boundary element method (BEM) is well-suited to resolving such flows, especially those with a low ratio of boundary to fluid regions [13,5,18]. The main advantage of the method is the dimensional reduction from a problem defined on the d-dimensional fluid region Ω to one defined on the $(d-1)$-dimensional fluid-solid boundary Γ. Such dimensional reduction results in more efficient employment of the collocation points and boundary elements when compared to conventional discretisation methods such as finite element and finite volume methods.

While conventional boundary element formulations use the mean fluid velocity to define such periodic flow problems, the mean pressure gradient is often the quantity measured experimentally, and so a boundary element formulation with pressure boundary conditions is desirable. To date, an efficient boundary element formulation for doubly-periodic two-dimensional Stokes flow with pressure boundary conditions has not been developed.

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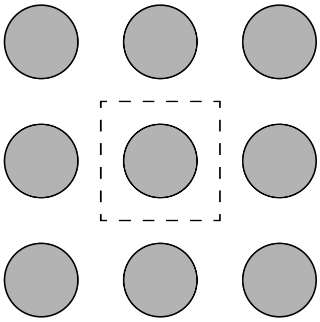

Fig. 1. A simple example of a 2D2P medium, where the dotted line represents the boundary of the unit cell.

Our motivation for seeking such a formulation is for use as a micro-scale model in a dual-scale modelling framework, such as one proposed by Alyaev et al. [3]. In this dual-scale framework, the macro-scale pressure gradient is used to specify boundary conditions on a micro-scale problem, and the mean velocity obtained by solving this problem is used to specify the macro-scale flux. Since such an algorithm would require the solution of many micro-scale problems, efficiency of the BEM solver is of key importance.

For a two-dimensional, doubly-periodic domain (2D2P; see Fig. 1), the prevailing BEM formulation in the literature [20] does not depend explicitly on the pressure gradient, and instead requires knowledge of the mean fluid velocity. The introduction of a nonzero mean pressure gradient by specifying pressure boundary conditions offers potential for a BEM formulation independent of an arbitrary mean velocity. Furthermore, the duality of the pressure and velocity formulations allows for a relationship, namely permeability, to be found between pressure gradient and mean velocity, independent of any fluid or flow parameters. Analytic solutions for Stokes flow past square and hexagonal arrays of circular, square and rounded hexagonal cylinders [22,28,29] can be used to test the BEM formulation, both with respect to pointwise velocity and to permeability.

A primary consideration is the efficiency of the method, especially in the calculation of the Green's function. The 2D2P Green's function [10,27] can be written as two quickly-converging sums, one over the physical lattice and one over the reciprocal lattice, by using the Ewald summation technique [8]. This technique includes a splitting parameter α that controls the trade-off between fast physical convergence and fast reciprocal convergence. In this sense, fast convergence refers to the rate at which terms in the sum decay, not the speed of a computational algorithm. For a fixed tolerance, the relevant numerical parameters are the splitting parameter and the truncation points of each sum. We refer to these as the convergence parameters, and note that specification of one of these defines the other two.

Values of α have been proposed that ensure the same rate of convergence between the sums [27] or a bias towards fast reciprocal convergence [10]. However, since the physical sum is much more computationally expensive to compute, giving bias towards fast physical convergence should increase computational efficiency, despite greatly increasing the requisite number of terms in the reciprocal sum.

Given the above motivation, the aim of the current work is to develop an efficient 2D2P BEM formulation for pressure-driven Stokes flow that calculates both the velocity field and permeability tensor, and avoids any dependence on the unknown mean fluid velocity. The accuracy of this formulation will be verified and its efficiency will be benchmarked.

In § 2, we outline the context of our work within the literature, especially with regards to the BEM and 2D2P Green's functions. We describe the current mean velocity-based BEM formulation and the current treatment of the convergence parameters. In § 3, we present dual formulations of the 2D2P BEM, derive the permeability by using the duality of these formulations, and detail an algorithm to approximate the optimal convergence parameters in the context of the BEM. In § 4, we exhibit the accuracy of the pressure-based formulation and investigate the numerical overheads necessary for each approach to the convergence parameters.

2. Background

2.1. Boundary element method

We begin by providing a brief overview of the context and history of the development of the BEM. For a more detailed account, refer to Cheng and Cheng [6].

Green [9] showed that for linear partial differential equations, it is possible to use fundamental solutions (Green's functions) to transform the differential equation of interest into an integral equation over the domain boundary. For the three-dimensional Laplace equation $\nabla^2\phi = 0$, defined in a region V, Green used the fundamental solution $1/|\mathbf{x} - \mathbf{x}_0|$ to derive the following identity [6]:

$$\phi(\mathbf{x}_0) = \frac{1}{4\pi} \iint\limits_{\partial V} \left[\frac{1}{r} \frac{\partial \phi(\mathbf{x})}{\partial n} - \phi(\mathbf{x}) \frac{\partial(1/r)}{\partial n} \right] \mathrm{d}S(\mathbf{x}), \tag{1}$$