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An exponential time-integrator scheme for steady and unsteady inviscid flows

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ABSTRACT

An exponential time-integrator scheme of second-order accuracy based on the predictor–corrector methodology, denoted PCEXP, is developed to solve multi-dimensional nonlinear partial differential equations pertaining to fluid dynamics. The effective and efficient implementation of PCEXP is realized by means of the Krylov method. The linear stability and truncation error are analyzed through a one-dimensional model equation. The proposed PCEXP scheme is applied to the Euler equations discretized with a discontinuous Galerkin method in both two and three dimensions. The effectiveness and efficiency of the PCEXP scheme are demonstrated for both steady and unsteady inviscid flows. The accuracy and efficiency of the PCEXP scheme are verified and validated through comparisons with the explicit third-order total variation diminishing Runge–Kutta scheme (TVDRK3), the implicit backward Euler (BE) and the implicit second-order backward difference formula (BDF2). For unsteady flows, the PCEXP scheme generates a temporal error much smaller than the BDF2 scheme does, while maintaining the expected acceleration at the same time. Moreover, the PCEXP scheme is also shown to achieve the computational efficiency comparable to the implicit schemes for steady flows.

1. Introduction

Significant progress has been made recently in the development of high-order spatial discretization methods in computational fluid dynamics (CFD), such as the discontinuous Galerkin (DG) [1–8], multi-moment constrained finite-volume (MCV) [9], flux reconstruction (FR) or correction procedure *via* reconstruction (CPR) method [10–12], and others [13–15]. These high-order techniques have exhibited a great potential as effective numerical solution methods amenable for efficient implementation on massively parallel high-performance computers. For complex geometries, an efficient solution, however, also depends on the availability of a fast time advancement solver. In contrast to a relative ubiquity of efficient techniques for spatial discretizations, efficient time-marching approaches for both steady and unsteady flows seem to be limited. Efficient time-integration approaches are thus the focus of the present work.

For unsteady flows, explicit methods, such as Runge–Kutta (RK) approaches are prevalent for their simplicity. However, with highly clustered nonuniform meshes, the Courant–Friedrichs–Lewy (CFL) condition can severely limit the local time-

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step size. The restriction due to the CFL condition is particularly acute for direct numerical simulation (DNS) and large-eddy simulation (LES) of turbulent flows, which usually require very fine grids of high aspect ratios in near-wall regions. Thus, the restriction due to the CFL condition becomes a critical bottleneck in computational efficiency for explicit time-marching schemes.

To enhance the computational efficiency of explicit time-marching schemes, it is desirable to relax or to remove the limitation of the CFL condition. To this end, a class of schemes based on the exponential time integration shows a great potential [16–29]. In contrast to usual explicit time-marching schemes, these schemes allow much larger time-step sizes while maintaining excellent numerical stability.

In explicit time-marching methods, information cannot propagate beyond the localities constrained by the CFL condition in each time step. In exponential time-marching methods, however, information is propagated to the entire computational domain instantaneously through the global Jacobian, similar to implicit methods, thus significantly alleviating the restriction on time-step size imposed by the CFL condition, if not eliminating it altogether. As mentioned previously, a variety of schemes based on the exponential integration have been developed already (cf., *e.g.*, [16–29]). While the basic idea of exponential integration has been adopted in the aforementioned methods, the existing algorithms differ from each other in some aspects. There are two types of exponential schemes depending on the treatment of the nonlinear term, *i.e.*, explicit and implicit. The classic ETD scheme is a typical explicit scheme (cf., *e.g.*, [16]), while the implicit integrator factor method is an implicit one (cf., *e.g.*, [21–23]), which can usually alleviate the stiffness due to the nonlinear term but requires nonlinear iterations in each time step.

While most of the exponential schemes are applied to specialized equations [16–27] with either scalar exponentials or constant matrix exponentials, such as the applications to semilinear parabolic equations [28,29], and relatively few are applied to practical CFD problems (cf., *e.g.*, [30–32]) with time-dependent full matrix exponential computations. There are some key issues, such as the computational efficiency for steady problems and the temporal accuracy for unsteady problems, have yet to be fully investigated. The overarching goal of the present work is to develop an efficient and time-accurate exponential scheme to solve multi-dimensional fluid dynamic equations. Specifically, we develop a second-order exponential time-integrator scheme to solve the Euler equations for steady and unsteady problems in both two and three dimensions, and assess its accuracy and computational efficiency by comparing with several well-known explicit and implicit approaches.

The remainder of this paper is organized as follows. Section 2 discusses the construction of a second-order exponential scheme based on the predictor–corrector methodology, denoted as PCEXP, and its efficient implementation through the Krylov method. Section 3 describes a linear stability and error analysis of PCEXP for a simple model equation in one dimension. Section 4 presents the application of PCEXP to the Euler equations discretized with a high-order DG method in space. Section 5 presents the numerical results of this work including three inviscid flow problems: (a) the transportation of an isentropic vortex in 2D with a constant velocity; (b) subsonic flow over a NACA0012 airfoil with a Mach number $\mathrm{Ma} = 0.63$; and (c) subsonic flow over a sphere in 3D with $\mathrm{Ma} = 0.3$. The numerical results obtained with PCEXP are compared with third-order Total Variation Diminishing Runge–Kutta scheme (TVDRK3), implicit backward Euler (BE), and second-order backward difference formula (BDF2). Finally, Section 6 summarizes and concludes this work. Appendix A provides the details of the Jacobian matrices.

2. Exponential time-integrator schemes

In this section, we first develop a predictor–corrector based the second-order exponential time-integrator scheme, and then discuss the efficient implementation through the Krylov method. We also carry out a linear stability analysis of the proposed scheme applied to a model equation in 1D to demonstrate its feasibility of time marching with large time steps.

2.1. Predictor–corrector exponential time-integrator scheme (PCEXP)

We start with the following semi-discrete system of autonomous ordinary differential equations which may be obtained from a spatial discretization:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{R}(\mathbf{u}), \tag{1}$$

where $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^K$ denotes the vector of the solution variables and $\mathbf{R}(\mathbf{u}) \in \mathbb{R}^K$ the right-hand-side term which may be the spatially discretized residual terms of the discontinuous Galerkin method used in this work. The dimension K is the degrees of freedom which can be very large for 3D problems. Without loss of generality, we consider $\mathbf{u}(t)$ in the interval of one time step, *i.e.*, $t \in [t_n, t_{n+1}]$.

We apply the term splitting method [24] to treat Eq. (1):

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{J}_n\mathbf{u} + \mathbf{N}(\mathbf{u}), \tag{2}$$

where the subscript n indicates the value evaluated at $t = t_n$, $\mathbf{J}_n$ denotes the Jacobian matrix $\mathbf{J}_n := \partial\mathbf{R}(\mathbf{u})/\partial\mathbf{u}|_{t=t_n} := \partial\mathbf{R}(\mathbf{u}_n)/\partial\mathbf{u}$, $\boldsymbol{u}_n := \mathbf{u}(t_n)$, and $\mathbf{N}(\mathbf{u}) := \mathbf{R}(\mathbf{u}) - \mathbf{J}_n\mathbf{u}$ denotes the remainder, which in general is nonlinear. Equation (2) admits the following formal solution: