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Well-posed and stable transmission problems

Jan Nordström *, Viktor Linders

Department of Mathematics, Computational Mathematics, Linköping University, SE-581 83 Linköping, Sweden

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ABSTRACT

We introduce the notion of a transmission problem to describe a general class of problems where different dynamics are coupled in time. Well-posedness and stability are analysed for continuous and discrete problems using both strong and weak formulations, and a general transmission condition is obtained. The theory is applied to the coupling of fluid-acoustic models, multi-grid implementations, adaptive mesh refinements, multi-block formulations and numerical filtering.

1. Introduction and motivation

In this paper we will introduce a general class of initial-boundary value problems coupled in time, which we refer to as *transmission problems*. This class includes any setting described by the following schematic:

1. The solution u is governed by the dynamics $\mathcal{D}_1$ from time t_1 to time t_2,
2. At t_2, the solution is subject to an operation $v = \mathcal{X}(u)$,
3. At later times, the solution v is governed by the (possibly different) dynamics $\mathcal{D}_2$.

Fig. 1 illustrates the above schematic. Central to this class of problems is the transmission operator $\mathcal{X}$, which we assume admits a matrix representation, but is otherwise left completely general.

Note that we consider a coupling procedure in time rather than space. Spatially coupled problems have been considered e.g. in [1] in the context of multi-physics problems and in [2–4] in the context of general conforming and non-conforming grids, and forms an integral part of finite element, discontinuous Galerkin and flux reconstruction algorithms. Spatially coupled problems typically must obey well-posedness or stability conditions that are strongly dependent on the nature of $\mathcal{D}_1$ and $\mathcal{D}_2$. In this paper, we will show that the temporal coupling involved in the transmission problem is in some sense independent of the dynamics involved, as long as $\mathcal{D}_{1,2}$ define two well-posed problems.

The formulation of the transmission problem is very general and consequently there are many practical applications that fit the framework. Examples considered in this paper include, with continuous time, the coupling in a fluid-acoustics problem, multi-grid techniques and adaptive mesh refinement. With discrete time, we exemplify with multi-block formulations for adaptive time-stepping and numerical filtering.

* Corresponding author.
E-mail addresses: jan.nordstrom@liu.se (J. Nordström), viktor.linders@liu.se (V. Linders).

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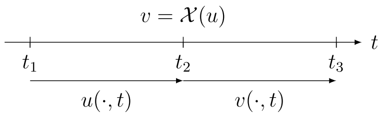

Fig. 1. Schematic of the transmission problem.

The aim of this paper is to obtain conditions for $\mathcal{X}$ under which the solution to the transmission problem is bounded by available data, in particular initial data available at time t_1; a prerequisite for well-posedness. However, boundedness of a solution depends on the norm in which it is estimated. This necessitates certain assumptions on the operators $\mathcal{D}_{1,2}$, and we will therefore confine the analysis to operators that are *semi-bounded* in a generalised L^2-norm [5,6]. In essence, this means that the transmission problem is amenable to analysis via the energy method.

We will consider continuous transmission problems where initial, boundary and coupling conditions are imposed either strongly, or weakly through so called lifting operators [7,8]. It will be shown that energy boundedness is equivalent to a certain condition relating the operator $\mathcal{X}$ and the norms in which the solution is estimated before and after the transmission time t_2. This *transmission condition* turns out to be independent of whether a strong or weak imposition is used. We will also discuss the implications of weighted norms on the transmission conditions and energy estimates.

Semi-discrete and fully discrete transmission problems will also be considered. In the latter cases, we utilise the theory of Summation-by-Parts (SBP) operators, first introduced in [9,10] to provide a means of obtaining stable finite difference procedures for the spatial discretisation of initial-boundary value problems. Since then, the SBP framework has been extended to methods outside the finite difference paradigm, including finite volume methods [11–13], spectral collocation, Galerkin and element methods [14–16], correction procedures via reconstruction [17] and temporal discretisations [18–21]. This opens the door to utilising energy arguments to analyse the stability of fully discrete numerical schemes, if initial, boundary and transmission conditions are imposed weakly [22,23].

It will be shown that all fully discrete transmission problems considered must satisfy a certain condition to obtain an energy estimate. This condition is completely analogous to the one obtained in the continuous setting. Throughout the paper we aim to keep the governing equations – continuous or discrete – as general as possible. Thus, the conditions derived will apply to a wide range of problems and numerical schemes, and may serve as an a priori test for the availability of an energy estimate in a given norm.

The remainder of the paper is structured as follows: In section 2 we introduce the notation and definitions required henceforth. The transmission problem is formally introduced in section 3 and a necessary and sufficient condition for energy boundedness is derived for both strong and weak formulations. In section 4, a discrete transmission problem is presented, and a stability analysis using a weak formulation is performed. We return to the transmission condition in section 5, show how to find weighted norms such that it is satisfied, and discuss its implications. A selection of applications illustrating the preceding theory are presented in section 6. Finally, conclusions are drawn in section 7.

2. Preliminaries

Before proceeding we introduce the necessary notation and definitions.

2.1. Semi-boundedness and well-posedness

Consider the initial-boundary value problem (IBVP)

$$\begin{aligned} u_t + \mathcal{D}(u) &= F, \quad t > 0, \quad x \in \Omega, \\ \mathcal{B}(u) &= g, \quad t \geq 0, \quad x \in \Gamma, \\ u &= f, \quad t = 0, \quad x \in \Omega \cup \Gamma, \end{aligned} \tag{1}$$

where Ω is an open d-dimensional region and $\mathcal{D}$ is a differential operator. The operator $\mathcal{B}$ defines a set of boundary conditions on the boundary Γ of Ω, and the functions F, g and f are given forcing, boundary and initial data.

For two functions u and v defined on Ω, we introduce the inner product and norm

$$(u, v)_P = \int_\Omega u^\top P v \mathrm{d}x, \quad \|u\|_P = (u, u)_P^{1/2}, \tag{2}$$

where P is a positive definite matrix whose dimension matches the number of variables contained in the vectors u and v. We will need the following two definitions [5,6].

Definition 1. Let $\mathbb{V}$ be the space of differentiable functions satisfying the boundary conditions $\mathcal{B}(v) = 0$. The differential operator $\mathcal{D}$ is *semi-bounded* if for all $v \in \mathbb{V}$, $\mathcal{D}(v)$ satisfies the inequality

$$(v, \mathcal{D}(v))_P \geq 0. \tag{3}$$