Journal of Computational Physics 364 (2018) 137–152

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Functionally-fitted energy-preserving integrators for Poisson systems ☆

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A R T I C L E I N F O

Article history:
Received 16 December 2017
Received in revised form 1 March 2018
Accepted 8 March 2018
Available online 13 March 2018

Keywords:
Poisson systems
Energy preservation
Functionally-fitted integrators

A B S T R A C T

In this paper, a new class of energy-preserving integrators is proposed and analysed for Poisson systems by using functionally-fitted technology. The integrators exactly preserve energy and have arbitrarily high order. It is shown that the proposed approach allows us to obtain the energy-preserving methods derived in [12] by Cohen and Hairer (2011) and in [1] by Brugnano et al. (2012) for Poisson systems. Furthermore, we study the sufficient conditions that ensure the existence of a unique solution and discuss the order of the new energy-preserving integrators.

1. Introduction

In this paper, we deal with the efficient numerical integrators for solving the Poisson systems (non-canonical Hamiltonian systems)

$$y'(t) = B(y(t))\nabla H(y(t)), \quad y(0) = y_0 \in \mathbb{R}^d, \quad t \in [0, T], \tag{1}$$

where the prime denotes $\frac{\mathrm{d}}{\mathrm{d}t}$, $B(y)$ is a skew-symmetric matrix which is not required to satisfy the Jacobi identity, $H(y)$ is a scalar vector field, and both are sufficiently smooth. It is assumed that the system (1) has a unique solution $y = y(t)$ defined for all $t \in [0, T]$. It is well known that the energy $H(y)$ is preserved along the exact solution $y(t)$, since

$$\frac{\mathrm{d}}{\mathrm{d}t}H(y(t)) = \nabla H(y(t))^{\intercal} y'(t) = \nabla H(y(t))^{\intercal} B(y(t))\nabla H(y(t)) = 0.$$

Numerical integrators that preserve $H(y)$ are usually called energy-preserving (EP) integrators, and the aim of this paper is to formulate and analyse novel EP integrators for efficiently solving Poisson systems.

When the matrix $B(y)$ is independent of y, d is an even number and

$$B = J = \begin{pmatrix} 0_{\frac{d}{2}} & I_{\frac{d}{2}} \\ -I_{\frac{d}{2}} & 0_{\frac{d}{2}} \end{pmatrix},$$

☆ The research is supported in part by the Alexander von Humboldt Foundation, by the Natural Science Foundation of Shandong Province (Outstanding Youth Foundation) under Grant ZR2017JL003, and by the National Natural Science Foundation of China under Grant 11671200.

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https://doi.org/10.1016/j.jcp.2018.03.015
0021-9991/

the system (1) is a canonical Hamiltonian system. There have been a lot of studies on numerical methods for this system, and the reader is referred to [14,17–19,25,31,33,35,36,38,41] and references therein. For canonical Hamiltonian systems, EP methods are an important and efficient kind of methods and many various of EP methods have been derived and studied in the past few decades, such as the average vector field (AVF) method (see, e.g. [9,10,28]), discrete gradient methods (see, e.g. [23,24]), Hamiltonian Boundary Value Methods (HBVMs) (see, e.g. [3,4]), EP collocation methods (see, e.g. [16]) and exponential/trigonometric EP methods (see, e.g. [21,27,32,34,39]).

Among these EP methods for solving $\dot{y} = J\nabla H(y)$, the AVF method has the simplest form, which was given by [24] as follows

$$y_1 = y_0 + h\int_0^1 J\nabla H(y_0 + \sigma(y_1 - y_0))d\sigma. \tag{2}$$

Quispel and McLaren in [28] revealed that this method is a B-series method. Hairer extended this second-order method to higher order schemes by introducing continuous stage Runge–Kutta methods [16]. However, because the dependence of the matrix $B(y)$ should be discretised in a different manner, Poisson systems usually require an additional technique. Therefore, the novel EP methods which are specially designed and analysed for Poisson systems are necessary. McLachlan et al. [24] discussed DG methods for various kinds of ODEs including Poisson systems. Cohen and Hairer in [12] succeeded in constructing arbitrary high-order EP schemes for Poisson systems and the following second-order EP scheme for (1) was derived

$$y_1 = y_0 + hB\big((1/2)(y_1 + y_0)\big)\int_0^1 \nabla H(y_0 + \sigma(y_1 - y_0))d\sigma. \tag{3}$$

Following the ideas of HBVMs, Brugnano et al. gave an alternative derivation of such methods and presented a new proof of their orders in [1]. EP exponentially-fitted integrators for Poisson systems were researched by Miyatake [26]. Based on discrete gradients, Dahlby et al. [13] constructed useful methods that simultaneously preserve several invariants in systems of type (1). Other multiple invariants preserving integrators are referred to [2,4,5,11,20] for example.

On the other hand, the functionally-fitted (FF) technology is a popular approach to constructing effective and efficient methods for solving differential equations. A FF method is generally derived by requiring it to integrate members of a given finite-dimensional function space X exactly. The corresponding methods are called as trigonometrically-fitted (TF) or exponentially-fitted (EF) methods if X is generated by trigonometrical or exponential functions. Using FF/TF/EF technology, many efficient methods have been constructed for canonical Hamiltonian systems including the symplectic methods (see, e.g. [6–8,15,29,30,37,40]) and EP methods (see, e.g. [22,27]). This technology has also been used successfully for Poisson systems in [26] and second- and fourth-order schemes were derived. In this paper, using the functionally-fitted technology, we will design and analyse novel EP integrators for Poisson systems. The new integrators can be of arbitrary order in a routine and convenient manner, and different EP schemes can be obtained by considering different function spaces. It will be shown that choosing a special function space allows us to obtain the EP schemes given by Cohen and Hairer [12] and Brugnano et al. [1].

This paper is organised as follows. In Section 2, we derive the EP integrators for Poisson systems. Section 3 is devoted to the implementation issues. The existence and uniqueness of the integrators are studied in Section 4 and their algebraic orders are discussed in Section 5. In Section 6, three EP schemes are presented as illustrative examples. Numerical experiments are implemented in Section 7, where we consider the Euler equation. The last section includes some conclusions.

2. Functionally-fitted EP integrators

Since the solution of (1) belongs to $\mathbb{R}^d$, we define a vector function space $Y = \text{span}\{\varphi_0(t), \dots, \varphi_{r-1}(t)\}$ on $[0, T]$ by (see [22])

$$Y = \left\{ w : w(t) = \sum_{i=0}^{r-1} \varphi_i(t) W_i,\ t \in [0, T],\ W_i \in \mathbb{R}^d \right\},$$

where the real functions $\{\varphi_i(t)\}_{i=0}^{r-1}$ are linearly independent and of $\mathbb{C}^1$ $([0, T] \to \mathbb{R})$. In this paper, choose a stepsize $h > 0$ and we consider the following two function spaces

$$Y_h = \text{span}\{\varphi_0(\tau h), \dots, \varphi_{r-1}(\tau h)\}, \quad X_h = \text{span}\left\{1, \int_0^{\tau h} \varphi_0(s)ds, \dots, \int_0^{\tau h} \varphi_{r-1}(s)ds\right\}, \tag{4}$$

where $\tau \in [0, 1]$. In this paper, τ is always denoted as a variable on the interval $[0, 1]$. We also note that the stepsize h is a positive parameter with $0 < h \le h_0 \le T$, where h_0 depends on the problem under consideration.

A projection given in [22] will be used in this paper and we summarise its definition as follows.