Journal of Computational Physics 362 (2018) 49–68

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

A hybrid framework for coupling arbitrary summation-by-parts schemes on general meshes

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ARTICLE INFO

Article history:
Received 21 September 2017
Received in revised form 7 February 2018
Accepted 10 February 2018
Available online 13 February 2018

Keywords:
Hybrid meshes
Numerical interfaces
Stability
Summation-by-parts
Finite volume
Finite difference

ABSTRACT

We develop a general interface procedure to couple both structured and unstructured parts of a hybrid mesh in a non-collocated, multi-block fashion. The target is to gain optimal computational efficiency in fluid dynamics simulations involving complex geometries. While guaranteeing stability, the proposed procedure is optimized for accuracy and requires minimal algorithmic modifications to already existing schemes. Initial numerical investigations confirm considerable efficiency gains compared to non-hybrid calculations of up to an order of magnitude.

1. Introduction

Computational fluid dynamics (CFD) has become a key technology in engineering innovation and science investigation. Despite significant advances, CFD as a design tool is still expensive when complex geometries are prevalent. In these circumstances, the optimal method of discretization as well as the optimal level of grid refinement often varies throughout the computational domain. Hence, the ability to combine general computational blocks in a modular fashion can be a powerful way of improving the efficiency in many types of simulations. For example, finite volume (FV) methods are still the most widely used in CFD due their applicability to unstructured meshes, even though at most second order spatial accuracy is routinely employed. Finite difference (FD) methods on the other hand represent the most efficient type of high order schemes, but their use is limited to structured meshes, and thus less complex geometries. The best attributes of each of these methods may however be harnessed in an innovative manner by developing a hybrid scheme. Such an approach involves utilizing both methods on a particular problem so as to ensure optimal computational efficiency. The key challenge to this involves interfacing the two schemes in a stable and efficient manner, with minimal modifications to either method.

A systematic approach to the formulation of energy stable schemes is provided by the concept of summation-by-parts (SBP) operators, originally introduced in [1], together with weak boundary [2] and interface conditions [3]. The summation-by-parts framework was for a long time restricted to high order accurate FD methods, gradually incorporating techniques to handle curvilinear and multi-block grids [4–6] in a stable manner. The initial hybrid development of interest to this paper began with the formulation a node-centered edge-based FV scheme in SBP form [7–9]. This was extended to hybrid FD-FV

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https://doi.org/10.1016/j.jcp.2018.02.018

couplings in [10–12]. The stability of this particular hybrid technique relies on a collocated FD-FV interface where the dual grid cells on the FV side are modified to match the integration weights and metric variables of the FD scheme.

Apart from FD and FV schemes, significant steps have also been taken towards unifying a much broader class of nodal methods with similar stability properties into the SBP framework. High order methods in this category include spectral [13, 14], discontinuous Galerkin (dG) [15], flux reconstruction [16,17], time integration [18–20] and space–time [21,22] methods. A recent related development to the hybrid FD-FV method described above is the use of so-called summation-by-parts preserving interpolation operators to couple non-collocated numerical domains, i.e. interfaces where the node distributions are non-matching, see [23–26]. So far this approach has been applied to non-collocated FD-FD as well as FD-dG [25] couplings, where different grid resolutions are used on each side of an interface.

In this paper we extend the concept of summation-by-parts preserving operators and formulate a hybrid framework for general SBP methods posed on arbitrary grids. We use the term general here in the sense that the formulation does not rely on the explicit application of either tensor products or curvilinear coordinate transformations. As such, the new framework is well suited for both structured and unstructured methods. To exemplify the new framework, we develop interface operators between FD blocks of arbitrary order involving curvilinear transformation, as well as between FD and FV blocks. Both the FD-FD and FD-FV couplings are combined with interpolation to allow for non-collocated node distributions, i.e. so that different grid resolutions can be used on both sides of a hybrid interface.

The rest of this article is organized as follows. In sections 2 and 3 we lay out a general theoretical framework for energy stable interface couplings between SBP schemes on arbitrary geometries. In section 4 we confirm that both the FV and high order FD methods conform to the general SBP framework. Once the theoretical foundation is in place, we proceed in section 5 to derive new FD-FD as well as FD-FV interface operators for curvilinear grids, including the case of non-collocated grid distributions. In section 6 we perform numerical calculations and study the convergence properties and efficiency of the new hybrid schemes. Finally, in section 7, we draw conclusions.

2. Summation-by-parts operators

The summation-by-parts methodology may be understood as a systematic approach to construct stable numerical schemes for linear problems. Stability for non-linear problems with smooth solutions then follows by application of the energy method to all frozen coefficient versions of the problem at hand, see e.g. [27] for a review of this technique. In the remainder, we will assume that the reader is familiar with the energy method for analyzing well-posedness and stability.

2.1. Continuous analysis

For simplicity of presentation, the analysis in this paper will be carried out for the advection–diffusion problem,

$$\begin{aligned} u_t + \mathbf{a}^T\nabla u &= \epsilon\nabla^T\nabla u, & \mathbf{x}\in\Omega\subset\mathbb{R}^r, && t>0 \\ \frac{1}{2}(\mathbf{a}^T\mathbf{n} - |\mathbf{a}^T\mathbf{n}|)u - \epsilon\mathbf{n}^T\nabla u &= g(t,\mathbf{x}), & \mathbf{x}\in\partial\Omega, && t>0 \\ u &= f(\mathbf{x}), & \mathbf{x}\in\Omega, && t=0 \end{aligned} \tag{1}$$

posed on an arbitrary domain $\mathbf{x}\in\Omega\subset\mathbb{R}^r$ in r space dimensions. In (1), $\mathbf{a}$ is a constant vector, $u(\mathbf{x},t)$ is the solution and we use well-posed Robin boundary conditions. Other forms of boundary conditions in (1) are also possible, including Dirichlet conditions [28]. However, since the main focus in this article is on interface conditions, it is sufficient for our needs to consider only one form of well-posed boundary conditions.

We introduce the continuous inner products for sufficiently smooth scalar functions ϕ and ψ,

$$(\phi,\psi)_\Omega = \int_\Omega \phi\psi\, dV,\ (\phi,\psi)_{\partial\Omega} = \oint_{\partial\Omega} \phi\psi\, dS,\ (\nabla\phi,\nabla\psi)_\Omega = \int_\Omega (\nabla\phi)^T(\nabla\psi)\, dV \tag{2}$$

where $\int_\Omega$ and $\oint_{\partial\Omega}$ are volume and surface integrals, respectively. The integration rules required for analyzing (1) with the energy method are,

$$(\phi,\mathbf{a}^T\nabla\psi)_\Omega = (\phi,\mathbf{a}^T\mathbf{n}\psi)_{\partial\Omega} - (\mathbf{a}^T\nabla\phi,\psi)_\Omega \tag{3}$$

$$(\phi,\nabla^T\nabla\psi)_\Omega = (\phi,\mathbf{n}^T\nabla\psi)_{\partial\Omega} - (\nabla\phi,\nabla\psi)_\Omega \tag{4}$$

where $\mathbf{n}$ is the outward pointing unit normal to the domain. With zero boundary data $g=0$, the energy method applied to (1) yields the (bounded) energy rate

$$\frac{\partial}{\partial t}\|u\|^2_\Omega + 2\epsilon\|\nabla u\|^2_\Omega = -(u,|\mathbf{a}^T\mathbf{n}|u)_{\partial\Omega}, \tag{5}$$

which proves well-posedness.