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On discretely entropy conservative and entropy stable discontinuous Galerkin methods

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ABSTRACT

High order methods based on diagonal-norm summation by parts operators can be shown to satisfy a discrete conservation or dissipation of entropy for nonlinear systems of hyperbolic PDEs [1,2]. These methods can also be interpreted as nodal discontinuous Galerkin methods with diagonal mass matrices [3–6]. In this work, we describe how use flux differencing, quadrature-based projections, and SBP-like operators to construct discretely entropy conservative schemes for DG methods under more arbitrary choices of volume and surface quadrature rules. The resulting methods are semi-discretely entropy conservative or entropy stable with respect to the volume quadrature rule used. Numerical experiments confirm the stability and high order accuracy of the proposed methods for the compressible Euler equations in one and two dimensions.

1. Introduction

Numerical simulations in engineering increasingly require higher accuracy without sacrificing computational efficiency. Because they are more accurate than low order methods per degree of freedom for sufficiently regular solutions, high order methods provide one avenue towards improving fidelity in numerical simulations while maintaining reasonable computational costs. High order methods which can accommodate unstructured meshes are desirable for problems with complex geometries, and among such methods, high order discontinuous Galerkin (DG) methods are particularly well-suited to the solution of time-dependent hyperbolic problems on modern computing architectures [7,8].

The accuracy of high order methods can be attributed in part to their low numerical dissipation and dispersion compared to low order schemes [9]. This accuracy has made them advantageous for the simulation of wave propagation [7,10]. However, while high order methods can be applied in a stable manner to linear wave propagation problems, instabilities are observed when applying them to nonlinear hyperbolic problems. This is contrast to low order schemes, whose high numerical dissipation tends to apply a stabilizing effect [11]. As a result, most high order schemes for nonlinear conservation laws typically require additional stabilization procedures, including filtering [7], slope limiting [12], artificial viscosity [13], and polynomial de-aliasing through over-integration [14]. Moreover, stabilized numerical methods can still fail, requiring user intervention or heuristic modifications to achieve non-divergent solutions.

For linear wave propagation problems, semi-discretely energy stable numerical methods can be constructed, even in the presence of curvilinear coordinates or variable coefficients [15–18]. This semi-discrete stability implies that, under a stable timestep restriction (CFL condition), discrete solutions do not suffer from non-physical growth in time. However, for non-

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linear systems of conservation laws, traditional methods do not admit theoretical proofs of semi-discrete stability. This was addressed for low order methods with the introduction of discretely entropy conservative and entropy stable finite volume schemes by Tadmor [19]. These schemes rely on a specific entropy conservative flux which satisfies a condition involving the entropy variables and entropy potential, and were extended to low order finite volume methods on unstructured grids in [20]. High order entropy stable methods were also developed for structured grids in [21] based on an entropy conservative essentially non-oscillatory (ENO) reconstruction.

The extension of entropy conservative schemes to unstructured high order methods was done much more recently for the compressible Euler and Navier–Stokes equations in [1,2] based on a spectral collocation approach on tensor product elements, which can also be interpreted as a mass-lumped DG spectral element (DG-SEM) scheme. The proof of entropy conservation relies on the presence of a diagonal mass matrix, the summation by parts (SBP) property [22], and a concept referred to as *flux differencing*. Similar entropy-stable schemes have been constructed for the shallow water and MHD equations [4,5,23]. Finally, high order entropy conservative and entropy stable schemes have been extended to unstructured triangular meshes in [24,6].

It is possible to construct energy preserving schemes for certain conservation laws based on split forms of conservation laws, which involve both conservative and non-conservative derivative terms. Split formulations have been shown to recover kinetic energy preserving schemes for the compressible Euler and Navier–Stokes equations under diagonal norm SBP operators [25–27]. Additionally, for dense norm and generalized SBP operators, stable schemes for Burgers' equation can be constructed based on the split form of the underlying equations [22,28,29]. However, entropy conservative and entropy stable schemes for the compressible Euler or Navier–Stokes equations do not correspond to split formulations [6], and (to the authors knowledge) the construction of unstructured high order entropy conservative and entropy stable schemes for these equations has required diagonal norm SBP operators.[1] We refer to DG methods with these properties as diagonal norm SBP-DG methods.

Appropriate diagonal norm SBP operators are straightforward to construct on tensor product elements based on a DG-SEM discretization. Diagonal-norm SBP operators can also be constructed for triangles and tetrahedra [31,32,6]; however, the number of nodal points for such operators is typically greater than the dimension of the natural polynomial approximation space, and the resulting diagonal norm SBP-DG operators do not correspond to any basis [32]. Furthermore, to the author's knowledge, appropriate point sets have only been constructed for $N \leq 4$ in three dimensions [33], and the construction of high order diagonal norm SBP-DG methods has not yet been performed for uncommon elements such as pyramids [34].

This work focuses on the construction of entropy conservative high order DG schemes for systems of conservation laws. In order to generalize beyond diagonal norm SBP-DG methods, we will consider DG discretizations using over-integrated quadrature rules with more points than the dimension of the approximation space, which are commonly used for non-tensor product elements in two and three dimensions [35]. These quadrature rules induce DG schemes which are related to dense norm and generalized SBP operators [36,28,37], for which discretely entropy stable schemes for the compressible Euler equations have not yet been constructed. We present proofs of discrete entropy stability using both a matrix formulation involving a "decoupled" SBP-like operator and continuous formulations involving projection and lifting operators. In both cases, the proofs rely only on properties which hold under quadrature-based integration. We also focus on ensuring discrete entropy stability for conservation laws which do not admit a nonlinearly stable split formulation.

The outline of the paper is as follows: Section 2 will briefly review the construction of entropy conservative diagonal norm SBP-DG methods on a single element. Section 4 will describe how to construct analogous entropy conservative methods on single element in a continuous setting. Section 5 will discuss extensions to multiple elements, including comparisons of different coupling terms and entropy stable fluxes. Finally, Section 6 presents numerical results which verify the high order accuracy and discrete entropy conservation of the proposed methods in one and two spatial dimensions.

2. Entropy stability for systems of hyperbolic PDEs

We will begin by reviewing continuous entropy theory. We consider systems of nonlinear conservation laws in one dimension with n variables

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0, \qquad \boldsymbol{u}(x,t) = (u_1(x,t), \ldots, u_n(x,t)), \tag{1}$$

where the fluxes $f(\boldsymbol{u})$ are smooth functions of the vector of conservative variables $\boldsymbol{u}(x,t)$. We are interested in systems for which there exists a convex entropy function $U(\boldsymbol{u})$ such that

$$U''(\boldsymbol{u})\boldsymbol{A}(\boldsymbol{u}) = \left(U''(\boldsymbol{u})\boldsymbol{A}(\boldsymbol{u})\right)^T, \qquad (\boldsymbol{A}(\boldsymbol{u}))_{ij} = \left(\frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial u_j}\right)_i, \tag{2}$$

where $\boldsymbol{A}(\boldsymbol{u})$ is the Jacobian matrix. For systems with convex entropy functions, one can define entropy variables $\boldsymbol{v} = U'(\boldsymbol{u})$. The convexity of the $U(\boldsymbol{u})$ guarantees that the mapping between conservative and entropy variables is invertible.

[1] Entropy stable high order finite element and DG methods which do not fall under the diagonal norm SBP-DG category have been proposed [30], but the proofs are often given at the continuous level, relying on exact integration or the chain rule, which do hold at the discrete level.