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A source term method for Poisson problems on irregular domains

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ABSTRACT

This paper presents a finite difference method for solving Poisson problems on a two-dimensional irregular domain. An implicit, level set representation of the domain boundary is assumed, as well as a Cartesian grid that is not fitted to the domain. The algorithm is based on a scheme for interface problems which captures the jump conditions via singular source terms. This paper adapts that method to deal with boundary value problems by employing a simple iterative process that simultaneously enforces the boundary condition and solves for an unknown jump condition. The benefit and novelty of this method is that the boundary condition is captured via easily implemented source terms. The system of equations that results at each iteration can be solved using a FFT-based fast Poisson solver. The scheme can accommodate Dirichlet, Neumann, and Robin boundary conditions. We first address the constant coefficient Poisson equation, and then extend the scheme to accommodate the variable coefficient equation. Numerical examples indicate second order accuracy (or close to it) for the solution. The method also produces useful gradient approximations, but with generally lower convergence rates.

1. Introduction

This paper presents a finite difference method for solving Poisson problems on a two-dimensional irregular domain. These problems arise in several applications including viscous incompressible flow [7,23,33] and shape identification [14,22]. A popular method of describing these domains, and for evolving their motion is the level set method [30,31,34]. With the level set method, the boundary is represented implicitly. This allows for complicated changes of topology to be represented in a natural way. In this setting it is advantageous to use a grid that is not fitted to the domain boundary, avoiding the computational cost of frequently re-fitting a conforming mesh as the geometry changes.

Numerical methods for Poisson problems on irregular domains have been studied for a long time, going back at least to the method of capacitance matrices in the early seventies [9]. More recently reference [27] proposed a method which assumes a Cartesian grid, can be used with fast Poisson solvers on a rectangular region, and is second order accurate. The solution is extended to the embedding rectangular region by solving a Fredholm integral equation. Reference [4] proposes a similar method for Laplace's equation with a Dirichlet boundary condition, and takes advantage of an accurate method for computing nearly singular integrals. Reference [28] uses similar ideas, combined with the fast multipole method. All of these methods assume an explicit, parameterized representation of the domain boundary.

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The Immersed Interface Method (IIM) [19–22,24] is a versatile technique that has been used to solve a wide array of problems, including the problem discussed here. A closely related method is the Explicit Jump Immersed Interface Method (EJIM) [40]. The IIM works with a Cartesian grid, and can be used with either an explicit parameterization, or an implicit level set description of the boundary. With this method, the standard finite difference scheme is modified near the boundary in order to achieve second or fourth order accuracy. The augmented IIM is the version of the IIM that is used to solve boundary value problems of the type considered in this paper. With this method augmented variables are introduced by orthogonally projecting grid points adjacent to the boundary onto the boundary itself, and an enlarged linear system of equations results. By projecting grid points from only one side of the boundary, the projected points are not clustered, and the potential problem of ill-conditioning is avoided. The resulting Schur complement system can be solved using an iterative method such as the Generalized Minimum Residual (GMRES) method. With this approach, it is possible to use a fast Poisson solver at each iteration, which is a benefit of the augmented variable technique. A least squares interpolation scheme is used to approximate the normal derivative of the solution on each side of the interface.

Reference [15] presents a finite difference method for the Dirichlet version of the problem considered here. The authors use a regular Cartesian grid, and by using conservative differencing of the second order fluxes, they achieve second order accuracy. They are able to handle variable coefficients, and their method can be used with adaptive mesh refinement. Their method requires an explicit representation of the boundary.

Reference [18] proposes a boundary integral method for the type of problem considered here. An integral equation is solved for a double layer density (for a Dirichlet boundary condition) or single layer density (for a Neumann boundary condition), and then the solution u is found by using the appropriate double or single layer integral representation. All of this is done within the level set framework, using an implicit representation of the boundary. What makes this possible is a novel method for computing boundary integrals in a level set framework. An advantage of this method is that because the solution is recovered via the integral representation, a uniform grid is not required, and so the method can be used with narrow banding, local level set, or adaptive grid methods. Also, this method handles the three types of boundary conditions addressed by our new algorithm, as well as mixed boundary conditions, where Dirichlet and Neumann boundary conditions are imposed on different portions of a single component of the boundary.

Reference [6] addresses the problems considered here as a special case of a more general interface problem. The authors use a Cartesian grid, and can handle both explicit and implicit representations of the boundary. Away from the boundary, the standard 5-point stencil is used, but near the boundary so-called virtual nodes are introduced. Their method is second order accurate.

The Ghost Fluid Method (GFM) was adapted to the Dirichlet version of the irregular domain problem in a series of papers [25,11,10,12]. The finite difference methods proposed in those papers assume a Cartesian grid, and can use a level set function to locate the boundary of the domain. Reference [25] deals with an interface problem where the elliptic operator may have a variable, discontinuous coefficient. The method is first order accurate, but captures jumps sharply, results in a simple linear algebra problem, and works in any number of spatial dimensions, since the interface is dealt with in a dimension-by-dimension fashion. In [11] this method was adapted to Poisson problems with Dirichlet boundary conditions. Without the interface coupling conditions required in the interface problems of [25], the authors were able to achieve second order accuracy. In [10], the authors devised a fourth order version of their scheme. The linear systems resulting from these methods are symmetric, making it possible to use relatively fast methods such as the Preconditioned Conjugate Gradient (PCG) method. Reference [12] incorporates an adaptive mesh refinement scheme into the GFM for these problems. A number of additional works have proposed methods closely related to, or derived from the GFM [16,26]. Papers that deal specifically with the Robin problem are [17] and [32].

Reference [13] proposes a method that starts with data specified on a uniform cartesian grid (or an adaptive quad-tree or oct-tree grid) but locally creates a Voronoi mesh near the boundary. It uses a fully implicit level set representation of the boundary and achieves second order accuracy in both two and three dimensions.

Reference [35] presents a method that extends the solution smoothly outside of the irregular domain onto a larger computational domain, and then uses Fourier spectral methods to produce an accurate solution, up to fourth order for Dirichlet problems and third order for Neumann problems. Reference [1] also achieves high accuracy using a solution extension method, along with an integral equation formulation and the fast multipole method. Both [35] and [1] employ an explicit representation of the boundary of the irregular domain.

In this paper we use a domain embedding method, along with an iterative approach that simultaneously determines an unknown jump condition and enforces the boundary condition of the original Poisson problem. Starting with an initial guess (of zero) for the undetermined portion of the jump data, we solve a discretization of the embedded Poisson problem. Next that solution (or its partial derivatives) is extrapolated from the interior of the domain onto a narrow strip containing the boundary to produce a new approximation of the unknown jump data, and then the process is repeated. At each iteration, the discrete problem is solved using a fast Poisson solver, and the process yields a fairly accurate solution after a reasonable number of iterations, see Fig. 1. The discretization of the embedded Poisson problem that is the basis for this process was shown in [39] to be second order accurate, and that accuracy seems to carry over to the new algorithm for the original Poisson problem with irregular boundary.

The “unknown jump data” concept underlying our method is also employed by the IIM, but with two important differences. First, although the unknown jump data can be viewed as an augmented variable, unlike the augmented IIM we do not