Journal of Computational Physics 360 (2018) 1–14

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

# A Kronecker product splitting preconditioner for two-dimensional space-fractional diffusion equations

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ARTICLE INFO

*Article history:*
Received 6 August 2017
Received in revised form 17 January 2018
Accepted 19 January 2018
Available online xxxx

*Keywords:*
Space fractional diffusion equation
Finite difference method
Krylov subspace method
Preconditioning
Matrix splitting

ABSTRACT

We study preconditioned iterative methods for the linear system arising in the numerical discretization of a two-dimensional space-fractional diffusion equation. Our approach is based on a formulation of the discrete problem that is shown to be the sum of two Kronecker products. By making use of an alternating Kronecker product splitting iteration technique we establish a class of fixed-point iteration methods. Theoretical analysis shows that the new method converges to the unique solution of the linear system. Moreover, the optimal choice of the involved iteration parameters and the corresponding asymptotic convergence rate are computed exactly when the eigenvalues of the system matrix are all real. The basic iteration is accelerated by a Krylov subspace method like GMRES. The corresponding preconditioner is in a form of a Kronecker product structure and requires at each iteration the solution of a set of discrete one-dimensional fractional diffusion equations. We use structure preserving approximations to the discrete one-dimensional fractional diffusion operators in the action of the preconditioning matrix. Numerical examples are presented to illustrate the effectiveness of this approach.



## 1. Introduction

We are interested in the following initial boundary value problem of two-dimensional space-fractional diffusion equation

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = d_+(x)\,{}_aD_x^{\alpha}u(x,y,t) + d_-(x)\,{}_xD_b^{\alpha}u(x,y,t), \\ \qquad\qquad + e_+(y)\,{}_cD_y^{\beta}u(x,y,t) + e_-(y)\,{}_yD_d^{\beta}u(x,y,t) + f(x,y,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,y,t) \in \Omega \times (0,T], \\ u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times [0,T], \\ u(x,y,0) = u_0(x,y), \quad (x,y) \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta < 2$ are the fractional derivative order, $f(x,y,t)$ is the source term, the nonnegative functions $d_\pm(x), e_\pm(y)$ are the diffusion coefficients and $\Omega = (a,b) \times (c,d)$. In addition, ${}_aD_x^{\alpha}$, ${}_xD_b^{\alpha}$, ${}_cD_y^{\beta}$ and ${}_yD_d^{\beta}$ are the left-sided and right-sided Riemann–Liouville fractional derivatives defined as follows

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https://doi.org/10.1016/j.jcp.2018.01.034

$$ {}_aD_x^{\alpha}u(x,y,t)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x\frac{u(\xi,y,t)}{(x-\xi)^{\alpha-1}}d\xi, $$

$$ {}_xD_b^{\alpha}u(x,y,t)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^b\frac{u(\xi,y,t)}{(\xi-x)^{\alpha-1}}d\xi, $$

$$ {}_cD_y^{\beta}u(x,y,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial y^2}\int_c^y\frac{u(x,\eta,t)}{(y-\eta)^{\beta-1}}d\eta, $$

$$ {}_yD_d^{\beta}u(x,y,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial y^2}\int_y^d\frac{u(x,\eta,t)}{(\eta-y)^{\beta-1}}d\eta, $$

where $\Gamma(\cdot)$ is the gamma function.

Fractional diffusion equations (FDEs) are currently considered one of the most widely used models for the description of anomalous diffusion. They have been proposed and investigated in many research fields, such as fluid mechanics, mechanics of materials, image processing, biology, finance, signal processing and control (see [1,3,13,17,25,31] and the references therein). With the increase of applications using FDEs, and analytical solutions are usually inaccessible for them, there is corresponding interest in the development and study of numerical methods for solutions of FDEs, see, for instance, [5,10,12,14,18,21–24,27,32,35,41,42].

Due to the nonlocal nature of the fractional differential operator, numerical solution of FDEs usually represents very intensive computational task. More specifically, numerical schemes for space FDEs typically yield dense and ill-conditioned linear systems. To limit the computational cost, fast algorithms based on fast Fourier transfrom (FFT) and preconditioning techniques have been developed [2,4,9,11,16,19,20,22,29,30,34,36–40].

The aim of this paper is the construction of a preconditioning method for the 2D FDE (1.1). Its implementation only requires in practice to define two local preconditioners for two matrices: obtained by discretizing two 1D FDEs. In this sense, we can recover the (already available) high performances of 1D problem preconditioners for the 2D FDE. The preconditioning is based on a formulation of the discrete problem that is shown to be the sum of two Kronecker products. The Kronecker product has an important property: if $U$ and $V$ are nonsingular, then $(U\otimes V)^{-1}=U^{-1}\otimes V^{-1}$ [15]. In other words, the inverse of a big matrix can be obtained from the inverses of much smaller matrices. Considering this property, the main idea in our preconditioners is to use one Kronecker product to approximate the summation of two Kronecker products. We obtain this Kronecker product through an alternating Kronecker product splitting of the coefficient matrix. The building blocks of this Kronecker product are discrete 1D fractional diffusion problems. For the discrete matrix derived from the 1D variable coefficient fractional diffusion equation, the authors of [11] proposed an effective structure preserving approximation. We will reuse this structure preserving approximation as the building block for our Kronecker product preconditioner. Note that the Kronecker product preconditioner is actually the splitting matrix of a class of splitting iteration methods. We demonstrate that the splitting method converges to the unique solution of the linear system and obtain the analytic values of the optimal iteration parameters when the eigenvalues of the discrete 1D fractional diffusion matrices are all real. Then we propose using a Krylov subspace method like GMRES to accelerate the convergence of the splitting iteration. However, the convergence result of the Kronecker product preconditioning method is still missing and that all the results are only numerics-oriented. We remark that this idea of Kronecker product splitting is similar to that in [6–8], but two linear systems are different.

This paper is structured as follows. In section 2 we state the matrix system arising in a finite difference discretization of the FDE (1.1). In section 3 we present an iterative method for this linear system based on an alternating Kronecker product splitting of the coefficient matrix. We analyze the convergence of the splitting iteration and consider its application in Krylov subspace methods. In Section 4, numerical examples are given to demonstrate the performance of the proposed preconditioner. Finally, concluding remarks are given in Section 5.

## 2. The discrete problem

Let us fix three positive integers $M$, $N_1$ and $N_2$, and define the following partition of $\Omega\times[0,T]$, i.e.,

$$ x_i=a+ih_1,\quad h_1=\frac{b-a}{N_1+1},\quad i=0,1,\ldots,N_1+1, $$

$$ y_j=c+jh_2,\quad h_2=\frac{d-c}{N_2+1},\quad j=0,1,\ldots,N_2+1, $$

$$ t_m=m\Delta t,\quad \Delta t=\frac{T}{M},\quad m=0,1,\ldots,M. $$