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An algorithm for the numerical evaluation of the associated Legendre functions that runs in time independent of degree and order

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A B S T R A C T

We describe a method for the numerical evaluation of normalized versions of the associated Legendre functions $P_\nu^{-\mu}$ and $Q_\nu^{-\mu}$ of degrees $0 \leq \nu \leq 1{,}000{,}000$ and orders $-\nu \leq \mu \leq \nu$ for arguments in the interval $(-1, 1)$. Our algorithm, which runs in time independent of ν and μ, is based on the fact that while the associated Legendre functions themselves are extremely expensive to represent via polynomial expansions, the logarithms of certain solutions of the differential equation defining them are not. We exploit this by numerically precomputing the logarithms of carefully chosen solutions of the associated Legendre differential equation and representing them via piecewise trivariate Chebyshev expansions. These precomputed expansions, which allow for the rapid evaluation of the associated Legendre functions over a large swath of parameter domain mentioned above, are supplemented with asymptotic and series expansions in order to cover it entirely. The results of numerical experiments demonstrating the efficacy of our approach are presented, and our code for evaluating the associated Legendre functions is publicly available.

1. Introduction

The associated Legendre functions arise in many contexts in applied mathematics and physics. They are perhaps most commonly encountered in connection with spherical harmonics, which are tensor products of associated Legendre functions and exponential functions. Among other things, the spherical harmonics are used to efficiently represent smooth functions given on the surface of the sphere, and in spectral methods for the solution of partial differential equations.

In this paper, we describe an algorithm for the numerical evaluation of versions of the associated Legendre functions of the first and seconds kinds. Following standard convention, we will denote by P_ν^μ the associated Legendre function of the first kind of degree ν and order μ, and by Q_ν^μ the associated Legendre function of the second kind of degree ν and order μ (see, for instance, Section 5.15 of [23] or Section 3.4 of [7] for definitions). Since the magnitudes of the associated Legendre functions are excessively large, even for parameters of relatively small magnitude, the normalized associated Legendre functions defined via

$$\bar{P}_\nu^\mu(x) = \sqrt{\left(\nu + \frac{1}{2}\right) \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)}}\, P_\nu^\mu(x) \tag{1}$$

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and

$$\bar{Q}_\nu^\mu(x) = \frac{2}{\pi}\sqrt{\left(\nu+\frac{1}{2}\right)\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}}\,Q_\nu^\mu(x) \tag{2}$$

are often used in lieu of P_ν^μ and Q_ν^μ. Our interest in associated Legendre functions stems principally from their connection with spherical harmonics, and for many calculations involving the spherical harmonics it is more convenient to work with the functions $\tilde{P}_\nu^\mu$ and $\tilde{Q}_\nu^\mu$ defined by the formulas

$$\tilde{P}_\nu^\mu(t) = \bar{P}_\nu^\mu(\cos(t))\sqrt{\sin(t)} \tag{3}$$

and

$$\tilde{Q}_\nu^\mu(t) = \bar{Q}_\nu^\mu(\cos(t))\sqrt{\sin(t)} \tag{4}$$

(the variable t corresponds to one of the coordinates in the standard parameterization of the unit sphere S^2). Our algorithm, which runs in time independent of the degree ν and order μ, allows for the evaluation of $\tilde{P}_\nu^{-\mu}$ and $\tilde{Q}_\nu^{-\mu}$ when $0 \le \nu \le 1{,}000{,}000$, $0 \le \mu \le \nu$ and $0 < t \le \frac{\pi}{2}$ (in particular, both ν and μ can take on noninteger values). It is a consequence of standard connection formulas (such as those appearing in Section 3.4 of [7]) that this suffices for the evaluation of $\bar{P}_\nu^\mu(x)$ and $\bar{Q}_\nu^\mu(x)$ for any $0 \le \nu \le 1{,}000{,}000$, $-\nu \le \mu \le \nu$ and $-1 < x < 1$.

1.1. Overview of our algorithm

The functions $\tilde{P}_\nu^{-\mu}$ and $\tilde{Q}_\nu^{-\mu}$ satisfy the second order linear ordinary differential equation

$$y''(t) + \left(\lambda^2 - \eta^2\csc^2(t)\right)y(t) = 0 \quad \text{for all } 0 < t < \frac{\pi}{2} \tag{5}$$

with $\lambda = \nu + \frac{1}{2}$ and $\eta^2 = \mu^2 - \frac{1}{4}$. By a slight abuse of terminology, we will refer to (5) as the associated Legendre differential equation. When $0 \le \mu \le \frac{1}{2}$, the coefficient of y in (5) is positive on the interval $\left(0, \frac{\pi}{2}\right)$, whereas when $\mu > \frac{1}{2}$ it is negative on the interval

$$\left(0, \arcsin\left(\frac{\eta}{\lambda}\right)\right) \tag{6}$$

and positive on

$$\left(\arcsin\left(\frac{\eta}{\lambda}\right), \frac{\pi}{2}\right). \tag{7}$$

It follows from these observations and well-known WKB estimates (see, for example, [9]) that when $\mu > \frac{1}{2}$, the solutions of (5) behave roughly like combinations of increasing or decreasing exponentials on (6) and are oscillatory on (7), whereas when $0 \le \mu \le \frac{1}{2}$, they are oscillatory on all of $\left(0, \frac{\pi}{2}\right)$. We will refer to the subset

$$\begin{aligned}\mathcal{O} = &\left\{(\nu,\mu,t) : \nu \ge 0,\ 0 \le \mu \le \min\left\{\nu, \frac{1}{2}\right\} \text{ and } 0 < t \le \frac{\pi}{2}\right\} \bigcup \\ &\left\{(\nu,\mu,t) : \nu > \frac{1}{2},\ \frac{1}{2} < \mu \le \nu \text{ and } \arcsin\left(\frac{\eta}{\lambda}\right) \le t \le \frac{\pi}{2}\right\}\end{aligned} \tag{8}$$

of $\mathbb{R}^3$ as the oscillatory region, and to the subset

$$\mathcal{N} = \left\{(\nu,\mu,t) : \nu \ge 0,\ \mu > \frac{1}{2} \text{ and } 0 < t < \arcsin\left(\frac{\eta}{\lambda}\right)\right\} \tag{9}$$

as the nonoscillatory region. When $\nu \gg \mu > \frac{1}{2}$, the solutions of (5) are highly oscillatory on (7), and when $\nu \ge \mu \gg \frac{1}{2}$, they behave roughly like combinations of rapidly decreasing and increasing exponentials on (6). Consequently, they cannot be effectively represented via polynomial expansions in the variables ν, μ and t on either of the sets $\mathcal{O}$ or $\mathcal{N}$, at least for large values of the parameters.

Nonetheless, the logarithms of certain solutions of (5) can be represented efficiently via polynomial expansions on the sets $\mathcal{N}$ and $\mathcal{O}$. This observation is related to the well-known fact that the associated Legendre differential equation admits a nonoscillatory phase function. Many special functions of interest posses this property as well, at least in an asymptotic sense [21,6]. However, the sheer effectiveness with which nonoscillatory phase functions can represent solutions of the general equation

$$y''(t) + \lambda^2 q(t)y(t) = 0 \quad \text{for all } a < t < b \tag{10}$$