Journal of Computational Physics 360 (2018) 74–92

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

# High performance computation of radiative transfer equation using the finite element method

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ARTICLE INFO

*Article history:*
Received 21 June 2017
Received in revised form 11 December 2017
Accepted 17 January 2018
Available online xxxx

*Keywords:*
Radiative transfer equation
Parallel computing
Discrete ordinate method
Vectorial finite element method

ABSTRACT

This article deals with an efficient strategy for numerically simulating radiative transfer phenomena using distributed computing. The finite element method alongside the discrete ordinate method is used for spatio-angular discretization of the monochromatic steady-state radiative transfer equation in an anisotropically scattering media. Two very different methods of parallelization, angular and spatial decomposition methods, are presented. To do so, the finite element method is used in a vectorial way. A detailed comparison of scalability, performance, and efficiency on thousands of processors is established for two- and three-dimensional heterogeneous test cases. Timings show that both algorithms scale well when using proper preconditioners. It is also observed that our angular decomposition scheme outperforms our domain decomposition method. Overall, we perform numerical simulations at scales that were previously unattainable by standard radiative transfer equation solvers.



## 1. Introduction

Energy transport via radiative transfer is significant for many fields of science and technology, especially when high temperatures are considered. The modeling of such radiative transfer phenomena has been gaining attention these last decades within diverse scientific fields, such as heat transfer [1,2], neutron transport [3,4], optical imaging [5,6], biomedical optics [7,8], astrophysics [9,10], radiative transport [11–13], etc.

Commonly, radiative transfer equation (RTE) is used to mathematically formulate the process of radiative transfer at mesoscopic/macroscopic scales [14]. For many modern applications, e.g., combustion in furnaces, solid rocket propulsion, gas turbine engine, heat exchange in concentrated solar power technologies, particle transport in nuclear reactors, and laser heating of materials to cite but a few, solving the three-dimensional RTE forms an essential requisite. Apart from considering the three spatial dimensions $(x, y, z)$, one also needs to incorporate two angular dimensions $(\theta, \phi)$, time $(t)$, and frequency $(\nu)$ dependencies for solving the full three-dimensional RTE, henceforth transforming it to a 7D problem. However, the physics considered in this article being restricted to monochromatic and steady-state, the 3D radiative transfer equation in participating media thus becomes a 5D problem. The main goal of this article is to design efficient numerical solvers for such problems.

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https://doi.org/10.1016/j.jcp.2018.01.027
0021-9991/

**Nomenclature**

| | | | |
|---|---|---|---|
| $t$ | time | $\omega_m$ | discrete solid angle/weight |
| $\boldsymbol{I}$ | radiative intensity | $x$ | $x$-coordinate |
| $\mathbb{I}$ | vectorial radiative intensity | $y$ | $y$-coordinate |
| $\mathbb{I}_m$ | discrete vectorial radiative intensity | $z$ | $z$-coordinate |
| $\widehat{I}$ | manufactured radiative intensity | $g$ | anisotropic coefficient |
| $I_b$ | blackbody emission | $N_a$ | number of angles |
| $I_m$ | discrete radiative intensity | $v$ | FE test function |
| $\boldsymbol{x}$ | Cartesian space coordinates | $\mathbb{V}$ | vectorial FE test function |
| $\boldsymbol{s}$ | direction vector | $\gamma$ | SUPG stabilizing coefficient |
| $\boldsymbol{s}_m$ | discrete direction vector | $\mathcal{V}$ | FE functional space |
| $\boldsymbol{s}_{\text{in}}$ | input direction vector | $\mathcal{W}^{N_a}$ | vectorial FE functional space |
| $\mathbb{S}$ | vectorial direction vector | $\varphi_i$ | FE basis function |
| $c$ | speed of light | $\mathcal{M}_{\boldsymbol{x}}$ | spatial mesh |
| $\Phi$ | scattering phase function | $\mathcal{M}_{\boldsymbol{s}}$ | angular mesh |
| $\kappa$ | absorption coefficient | $n_p$ | number of processes |
| $\sigma_s$ | scattering coefficient | $\mathbf{A}$ | global stiffness matrix |
| $\beta$ | extinction coefficient | $\mathbf{A_{i,i}}$ | diagonal submatrix |
| $T$ | temperature | $\mathbf{A_{i,j}}$ | off-diagonal submatrix |
| $\mathcal{S}^{n-1}$ | angular domain | $D$ | radiative density |
| $\mathcal{D}$ | spatial domain | $\widehat{D}$ | exact radiative density |
| $\partial\mathcal{D}$ | spatial domain boundary | $c_1$ | algorithmic constant |
| $N_v$ | number of mesh nodes | $h_{\boldsymbol{x}}$ | spatial mesh length |
| $\boldsymbol{n}$ | outward unit normal vector | $h_{\boldsymbol{s}}$ | angular mesh length |
| $\nu$ | frequency | $N_e$ | number of mesh elements |
| $\theta$ | zenith angle | $I_{\text{in}}$ | input radiative intensity |
| $\phi$ | azimuthal angle | d.o.f. | degree of freedom |
| $\Theta$ | phase–weight scattering matrix | nnz | number of nonzeros |
| $\omega$ | albedo | | |

In the past, a vast range of numerical methods have been applied for the solution of the RTE. Each of these methods claims to have one or the other advantage over the others. These methods are divided into two main groups: physics-based (stochastic) approaches, and mathematics-based (deterministic) discretization approaches. Among stochastic approaches, the Monte Carlo method is frequently used for solving radiative transfer. This choice is supported by diverse and huge amount of research [15–20]. In principle, such a method could be applicable to any problem, since it only involves following all the scattering and absorption events of photon packets inside the computational domain. Naturally, due to its iterative nature, such a process may become very time consuming when considering a three dimensional and/or an optically thick medium. This comes from the fact that photon packets need to be scattered several thousand times in such domains. However, the main advantage of this method is that it is very versatile and simple to set up, at least for simple geometries.

Alternatively, deterministic approaches consist in approximating the radiative transfer equation by mathematical means. Since this equation is expressed both in space $(x, y, z)$ and angles $(\theta, \phi)$, the discretization needs to be applied to all coordinates. Firstly, concerning the angular discretization, the $P_N$ method, also known as the spherical harmonics method, is one of the most commonly used methods, probably due to its easy implementation. This method, which was first introduced in [21], uses an orthogonal basis to express the harmonics on the unit sphere. Although this method has been applied extensively in radiative transfer [22–24], it lacks accuracy for consistent solutions in optically thin or semi-transparent media. Another approach consists in discretizing the angular space based on a quadrature rule. Introduced by Chandrasekhar [25], this so-called discrete ordinate method (DOM) consists in solving a set of semi-discretized RTEs, the weights associated to the discrete ordinates (angular discretizations) are given by a quadrature rule [26,27]. As suggested in [28], in this paper a unit sphere is partitioned and the weights associated to the angular discretizations are given by the measure of the related solid angles. One of the reasons for choosing this kind of discretization for the DOM being that it is consistent with the specular reflection treatment previously developed by the authors [29]. In certain situations, the DOM may be prone to ray effects and false scattering, cf. [30,31], leading to inaccurate solutions. Apart from these specific cases, the DOM remains the reference method when associated to an efficient numerical scheme.

Secondly, concerning the spatial coordinates, RTE can be discretized using the finite difference method, the method of short characteristics, the finite volume method, and the finite element method. Simplest among these are the finite difference method [32] and the method of short characteristic [33]. These are typically used for regular structured meshes and simple geometries. Among all, the finite volume method (FVM) is the most commonly used method. Recently, the review paper [34] listed the advances of the FVM for solving radiative transfer problems. The finite element method (FEM) for spa-