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An energy-stable method for solving the incompressible Navier–Stokes equations with non-slip boundary condition

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ABSTRACT

We introduce a stable method for solving the incompressible Navier–Stokes equations with variable density and viscosity. Our method is stable in the sense that it does not increase the total energy of dynamics that is the sum of kinetic energy and potential energy. Instead of velocity, a new state variable is taken so that the kinetic energy is formulated by the L^2 norm of the new variable. Navier–Stokes equations are rephrased with respect to the new variable, and a stable time discretization for the rephrased equations is presented.
Taking into consideration the incompressibility in the Marker-And-Cell (MAC) grid, we present a modified Lax–Friedrich method that is L^2 stable. Utilizing the discrete integration-by-parts in MAC grid and the modified Lax–Friedrich method, the time discretization is fully discretized. An explicit CFL condition for the stability of the full discretization is given and mathematically proved.

1. Introduction

Navier–Stokes equations are one of the most successful descriptions of natural phenomena. The equations help us to understand the internal working principle of fluid flows and predict the near future of fluid flows. Fluid flow is a primitive physics motion that interacts with many other physics motions, and the applications of Navier–Stokes equations range in numerous areas such as weather forecasting [22,16], physics-based animation [2,19], and computer-aided design [13,14].

Due to the ubiquitous and huge importance of fluid flows on the nature, Navier–Stokes equations have been intensively studied in mathematics, sciences and engineering, however the equations are still in the list of the unsolved Millennium problems [5]. We do not have a concrete theory for the well-posedness of their analytic solution yet. Another quest for Navier–Stokes equations is to generate a discrete solution that approximates presumably the seemed-to-exist analytic solution. From Chorin's projection method [6], there have been many successful methods for the approximation such as the works of Bell et al. [1], Kim and Moin [15], and Brown et al. [4]. For finite difference approximation, Weinan and Liu [26,27,25] gave the convergence analysis and the error estimates of the projection method. They showed that the numerical treatment of boundary conditions in pressure and intermediate velocity does not affect to the accuracy of the velocity in the Navier–Stokes equations. For finite element approximations, Guermond and Quartapelle [10] provided the error bounds for velocity and pressure on incremental/non-incremental fractional methods. And Codina [7] analyzed the stability of pressure on the method. They showed that pressure stability can be controlled by restricting the time step size. The methods work

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well in most benchmark problems. In some cases, there is a need to adjust parameters of CFL conditions. Such uncertainty on the parameters may puzzle the users in practice, especially in real-time simulations.

In this article, we intend to develop a numerical method that is stable and accurate without any uncertainty on parameters. Our method is stable in the sense that it does not increase the total energy. Throughout this paper, we consider the following incompressible Navier–Stokes equations that include the gravitational force.

$$\begin{cases} \rho\left(U_t + U\cdot\nabla U\right) &= -\nabla p + \nabla\cdot\left(\mu\left(\nabla U + \nabla U^T\right)\right) + \rho g\cdot e_y \text{ in } \Omega \\ \nabla\cdot U &= 0 \quad \text{in } \Omega \\ U &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

Here ρ and μ are variable density and viscosity, and $\rho g\cdot e_y$ is the gravitational force. The total energy of the fluid is the sum of kinetic energy and potential energy. The total energy decreases as time advances as shown below. During the derivation, we use the mass conservation $D\rho/Dt = \rho_t + U\cdot\nabla\rho = 0$ and the formula of shape derivative [21] with no-slip boundary condition $U|_{\partial\Omega} = 0$.

$$\begin{aligned} \frac{d}{dt}\left[\int_\Omega \frac{1}{2}\rho U^2 - \rho g y\, dV\right] &= \int_\Omega \rho U\cdot\frac{DU}{Dt} - \rho g\frac{Dy}{Dt}\, dV \\ &= \int_\Omega U\cdot\left(-\nabla p + \nabla\cdot\left(\mu\left(\nabla U + \nabla U^T\right)\right) + \rho g\cdot e_y\right) - \rho g U\cdot e_y\, dV \\ &= -\frac{1}{2}\int_\Omega \mu\left(\nabla U + \nabla U^T\right) : \left(\nabla U + \nabla U^T\right) dV \le 0 \end{aligned} \tag{2}$$

Most numerical methods approximate the nonlinear term in equation (1) explicitly, and the other terms implicitly. Thus the approximation consists of two steps: one is to solve a convection equation and the other is to solve the Stokes' equations with gravity force. One can ensure the overall stability, when both steps are stable. The first step does not involve gravity, thus its stability is in terms of kinetic energy, the square of L^2 norm of $\sqrt{\rho}U$. In the explicit discretization of convection, the velocity field is given from the previous time steps. In the manner of frozen coefficients [23], the first step is to solve the following linear convection.

$$f_t + U\cdot\nabla f = 0 \qquad \text{in } \Omega \tag{3}$$

Due to the nonslip boundary condition $U|_{\partial\Omega} = 0$, the domain Ω is closed under motion, and we have the following L^2 stability on f.

$$\begin{aligned} \frac{d}{dt}\left[\int_\Omega \frac{1}{2} f(x,t)^2\, dV\right] &= \int_\Omega f f_t dV = -\int_\Omega fU\cdot\nabla f\, dV \\ &= \int_\Omega f^2\nabla\cdot U + fU\cdot\nabla f\ dV \quad \left(\because = -\int_\Omega fU\cdot\nabla f\, dV\right) \\ &= \frac{1}{2}\int_\Omega f^2\nabla\cdot U\ dV \quad = 0 \quad (\because \nabla\cdot U = 0) \end{aligned} \tag{4}$$

We note that the linear convection of equation (3) needs to be solved to satisfy the L^2 stability to ensure the overall stability of Navier–Stokes solver and the L^2 stability stems from the incompressibility condition. For solving the linear convection, there are canonical methods such as ENO method [12,20], WENO method [18], and central method [17]. The canonical methods were designed to solve the general convection equation alone, and do not strictly satisfy the L^2 stability when the given velocity is incompressible. A direct use of a canonical method without modification cannot lead to the stability of total energy, unless one figures out the net sum of the increased energy of the convection and the decreased energy amount of the Stokes' solver.

Our advection scheme is merely first order accurate, and should be inferior to many state-of-the-arts algorithms such as ENO, WENO, and CIP. The algorithms produce better results than our scheme in volume conservation and resolving many details, however their L^2-stability combined with Hodge projection has not been proved yet as far as we know. Our advection scheme was specially designed to cope with Hodge projection in MAC grid, producing the overall L^2-stability.

We introduce in section 2 a modified Lax–Friedrich (LF) method that is L^2 stable when the incompressible velocity field is given in the Marker-and-Cell (MAC) grid [11]. The convection in Navier–Stokes equations is solved by the modified LF and the Stokes' equations are solved. The overall stability is proved in Theorems 2 and 3.