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Enforcing the Courant–Friedrichs–Lewy condition in explicitly conservative local time stepping schemes

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ABSTRACT

An optimally efficient explicit numerical scheme for solving fluid dynamics equations, or any other parabolic or hyperbolic system of partial differential equations, should allow local regions to advance in time with their own, locally constrained time steps. However, such a scheme can result in violation of the Courant–Friedrichs–Lewy (CFL) condition, which is manifestly non-local. Although the violations can be considered to be "weak" in a certain sense and the corresponding numerical solution may be stable, such calculation *does not guarantee the correct propagation speed for arbitrary waves*. We use an experimental fluid dynamics code that allows cubic "patches" of grid cells to step with independent, locally constrained time steps to demonstrate how the CFL condition can be enforced by imposing a constraint on the time steps of neighboring patches. We perform several numerical tests that illustrate errors introduced in the numerical solutions by weak CFL condition violations and show how strict enforcement of the CFL condition eliminates these errors. In all our tests the strict enforcement of the CFL condition does *not* impose a significant performance penalty.

1. Introduction

In a seminal paper [1] Richard Courant, Kurt Friedrichs, and Hans Lewy showed that any explicit numerical scheme for solving fluid dynamics equations – or any other parabolic or hyperbolic system of partial differential equations (PDEs) – can be *stable and converges* to the correct solution only if it satisfies what we now call the Courant–Friedrichs–Lewy (CFL) condition: *"The full numerical domain of dependence must contain the physical domain of dependence"* [2].

The CFL condition implies an upper limit on the local time step of a given resolution element in the explicit numerical scheme:

$$\Delta t \le C_{\rm CFL}\frac{\Delta x}{v}, \tag{1}$$

where v is the local maximum wave speed in the resolution element, but the inverse is not necessarily true – the local time step constraint does not guarantee that the CFL condition is satisfied. The CFL condition is stricter, since it is *non-local*:

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the physical domain of dependence also includes waves that can originate somewhere else in the domain and still reach a given location in time Δt.

The simplest way to ensure that the CFL condition is satisfied is to enforce it globally using the largest wave speed and the corresponding smallest time step in the entire computational volume. However, when the allowed time steps vary widely across the computational volume, this approach can be extremely inefficient because the vast majority of resolution elements are forced to advance with a much shorter time step than allowed by equation (1).

In the Adaptive Mesh Refinement (AMR) schemes this inefficiency is partly overcome by allowing for larger cells in regions where high resolution and small time steps are not needed. AMR implementations often use "graded time-stepping" [3,4], whereby the time step is reduced by a fixed factor ξ_t at each subsequent spatial refinement level, with the spatial cell size decrease by a factor of ξ_s and $\xi_t = \xi_s = 2$ being a common choice. The graded time-stepping scheme, however, does *does not* eliminate the inefficiency, because the time step in each resolution element still depends on the most restricting time step anywhere else in the solution. For example, if the time step in the most restricting resolution element is decreased, all time steps in all other resolution elements need to be decreased as well.

A more efficient approach would be to allow individual resolution elements, or small localized groups of resolution elements, to advance with their own time steps set by the *local* constraint (1). Such local time stepping can also make parallelization and load balancing more flexible by reducing dependencies between separate domains advanced in parallel. Performance gains and the ease of achieving them likely depend on the specific computational problem and the subject field. This approach does appear promising in cosmological and galaxy formation simulations, where the fraction of volume that requires high resolution and small steps tends to be very small.

Local time-stepping (LTS) is a relatively new approach in computational physics, although some early attempts to implement it in numerical schemes date to previous century (c.f. [5–10]). Nevertheless, there already exist several thousands research papers discussing various applications of local time stepping, from multi-rate ODE solvers to full PDE schemes with mesh refinement. While giving a full review of this rapidly developing field is beyond the scope of this paper, several excellent recent reviews provide a comprehensive coverage of the current state of the art [11–16].

In this paper we focus on a subset of the LTS schemes that are explicitly conservative (i.e. maintain the conservation of physical quantities to the machine precision by appropriately tracking and exchanging fluxes between interfaces).

The local time step constraint (1) can be derived analytically for many simple numerical schemes as a requirement for (linear) numerical stability. One can therefore imagine a numerical scheme in which each resolution element steps in time with its own local maximally allowed time step (1). Such a scheme does not necessarily satisfy the CFL condition and is not guaranteed to be numerically stable for non-linear solutions. Numerical stability can be assured by enforcing the so-called "update criterion" (also sometimes called "evolve condition") [17–21] in addition to the local time step constraint (1). The update criterion forbids resolution elements or local domains to advance beyond the next time moment of any of its neighbors:

$$t_i^{n+1} \le \min\left\{t_j^{n+1}\right\},\ j \in N_i, \tag{2}$$

where N_i is the set of neighbors for the resolution element i. However, as we discuss below, the update criterion does not, in general, guarantee that the CFL condition is satisfied, and a more stringent condition that we derive below needs to be enforced to guarantee that the CFL condition is satisfied exactly at all times.

Our approach is, perhaps, most similar to that of [22,23], who introduced a buffer region between parts of the solution stepping with different local time steps. However, we argue in the following section that this is necessary, but not sufficient for the strict enforcement of the CFL condition. The latter also requires the capability to reduce the time step of the slower part of the solution "mid-step" in response to the evolving fast part of the solution.

2. Enforcing the CFL condition

In this paper, we use examples based on an experimental fluid dynamics code that follows the evolution of the Euler equations on a uniform grid using a conservative Godunov-type scheme. The grid is split into patches – cubic groups of cells that share the same time step – for efficiency and ease of implementation. Neighboring patches exchange fluxes and other relevant information, so that the global solution is obtained over the entire domain. Although we present results for this specific implementation of fluid dynamics solver, we believe the conclusions we draw are general and do not depend on the patch size or even shape. Our conclusions also equally apply to an adaptively refined grid.

In Fig. 1 we illustrate the potential numerical artifacts that can arise if only the local time step constraint is imposed without the strict enforcement of the CFL condition. The sketch shows the "space-time diagram" for three grid "patches," along with grid cells (vertical dotted lines). The local time step constraint in patch 0 is 4 times stricter than in the other two patches. Thus, patch 0 makes 4 time steps, while the other two make just one step. The green lines show characteristics of the PDE solution. Here, for illustration, we assume that the scheme uses $C_{\rm CFL} = 1$; for $C_{\rm CFL} < 1$ a characteristic would traverse a $1/C_{\rm CFL}$ fraction of a cell. In all these cases, characteristics of the PDE propagate the length of one cell or less.

Problems arise when a wave crosses the boundary between the two patches with different time steps. A wave following the red world line starts at space-time point A, reaches the boundary between patches 0 and 1 at space-time point B, but