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A new multigrid formulation for high order finite difference methods on summation-by-parts form

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ABSTRACT

Multigrid schemes for high order finite difference methods on summation-by-parts form are studied by comparing the effect of different interpolation operators. By using the standard linear prolongation and restriction operators, the Galerkin condition leads to inaccurate coarse grid discretizations. In this paper, an alternative class of interpolation operators that bypass this issue and preserve the summation-by-parts property on each grid level is considered. Clear improvements of the convergence rate for relevant model problems are achieved.

1. Introduction

The multigrid method is a convergence acceleration technique that improves iterative solvers using grid coarsening [1,2]. This methodology leads to improved convergence for elliptic problems in a straightforward way [2,3], while partial differential equations with a dominant hyperbolic character are often handled with additional artificial dissipation [4,5]. Multigrid methods are used in various branches of applied mathematics and engineering, such as electromagnetics [6], magnetohydrodynamics [7] and fluid dynamics [8].

Two important components in multigrid methods are the restriction and prolongation operators which transfer the information between grids. Typically these operators are based on linear interpolation procedures, regardless of the accuracy of the discretization [3]. Although this is a natural choice for low order schemes, it may be inappropriate for high order ones. In this work we make use of a general relation between prolongation and restriction operators which was originally proposed in a different context [9]. The resulting interpolation operators lead to a consistent approximation at the boundaries and guarantee that the formal order of accuracy of the original scheme is retained at the interior nodes on the coarse grids.

By applying the specific grid transfer operators to Summation-by-Parts (SBP) formulations with Simultaneous-Approximation-Terms (SATs) [10] weakly imposing the boundary conditions, we arrive at a multigrid method with provable energy stable discretizations on each grid level. Moreover, the procedure allows for the introduction of the stable and accurate artificial viscosity procedure proposed in [11]. The resulting improved convergence of the new multigrid method is exemplified for several linear model problems.

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The rest of this paper is organized as follows: in Section 2 the main features of the two-level multigrid algorithm are presented. Section 3 introduces the SBP-SAT technique for high order finite difference discretizations. Section 4 deals with the construction of multigrid algorithms for linear problems using the new class of interpolation operators. In Section 5 we compare the effects of different prolongation and restriction operators on the multigrid convergence. Conclusions are drawn in Section 6.

2. The multigrid algorithm

Consider the following steady-state problem:

$$\begin{aligned} Lu &= f, \text{ in } \Omega, \\ Hu &= g, \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where L is a differential operator, H is a boundary operator, f and g are given functions, and Ω is the domain with boundary $\partial\Omega$. The boundary conditions are assigned in a way such that (1) is well-posed [12,13].

Remark 2.1. In this paper, we only consider linear operators L in (1).

The construction of a two-level multigrid scheme [2,3] for solving (1) consists of the following four steps:

1. Fine-grid discretization;
2. Error smoothing;
3. Coarse-grid correction;
4. Fine-grid update.

In the following sections, we will outline the main features of these four steps.

2.1. Fine-grid discretization

Consider a grid Ω_1, here called the *fine grid*, on Ω. A discrete problem associated to (1) on the fine grid Ω_1 has the general form

$$L_1\mathbf{u} = \mathbf{F}, \tag{2}$$

where L_1 is a discrete version of the operator L in (1) which also includes the boundary conditions. The vector $\mathbf{F}$ is a grid function which approximates f on the nodes of Ω_1, augmented with boundary data $\mathbf{g}$, and $\mathbf{u}$ is an approximate solution to (1). Typically Ω_1 has many nodes, and it is expensive to solve (2) directly. Furthermore, the discrete operator L_1 is assumed to be invertible and have eigenvalues with strictly positive real parts.

2.2. Error smoothing

An error smoothing procedure is required prior to grid coarsening. First, we consider marching towards the solution to (2) in pseudo time from an initial guess $\mathbf{u}^{(0)}$ by solving

$$\begin{aligned} \mathbf{w}_\tau + L_1\mathbf{w}(\tau) &= \mathbf{F}, \quad 0 < \tau < \Delta\tau, \\ \mathbf{w}(0) &= \mathbf{u}^{(0)}, \end{aligned} \tag{3}$$

for $\Delta\tau > 0$ called the *smoothing step*. The solution to (3) is

$$\mathbf{w}(\Delta\tau) = e^{-L_1\Delta\tau}\mathbf{u}^{(0)} + \left(I_1 - e^{-L_1\Delta\tau}\right)L_1^{-1}\mathbf{F}, \tag{4}$$

where I_1 indicates the identity matrix on Ω_1. If the eigenvalues of L_1 have strictly positive real parts, $\|\mathbf{w}(\Delta\tau) - \mathbf{u}\| < \|\mathbf{u}^{(0)} - \mathbf{u}\|$ for any norm.

More generally from (4), we may define a smoothing technique for (2) as

$$\begin{aligned} \mathbf{w}^k &= S\mathbf{w}^{k-1} + (I_1 - S)L_1^{-1}\mathbf{F}, \ k = 1, \ldots, \nu, \\ \mathbf{w}^0 &= \mathbf{u}^{(0)}, \end{aligned} \tag{5}$$

where the exponential smoother $S_{\text{exp}} = e^{-L_1\Delta\tau}$ yields the pseudo time-marching procedure in (4). After ν steps, the iterative method (5) leads to

$$\mathbf{w} = S^\nu\mathbf{u}^{(0)} + \left(I_1 - S^\nu\right)L_1^{-1}\mathbf{F}. \tag{6}$$