



# An integral equation-based numerical solver for Taylor states in toroidal geometries

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## ABSTRACT

We present an algorithm for the numerical calculation of Taylor states in toroidal and toroidal-shell geometries using an analytical framework developed for the solution to the time-harmonic Maxwell equations. Taylor states are a special case of what are known as Beltrami fields, or linear force-free fields. The scheme of this work relies on the generalized Debye source representation of Maxwell fields and an integral representation of Beltrami fields which immediately yields a well-conditioned second-kind integral equation. This integral equation has a unique solution whenever the Beltrami parameter  $\lambda$  is not a member of a discrete, countable set of resonances which physically correspond to spontaneous symmetry breaking. Several numerical examples relevant to magnetohydrodynamic equilibria calculations are provided. Lastly, our approach easily generalizes to arbitrary geometries, both bounded and unbounded, and of varying genus.

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## 1. Introduction

A wide range of astrophysical and laboratory plasmas are in force-free equilibria [2,27,51,55] where the magnetic field  $\mathbf{B}$  satisfies

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0} \quad (1.1)$$

This immediately implies that the current density  $\mathbf{J} = (\nabla \times \mathbf{B})/\mu_0$  is parallel to the magnetic field, i.e. there exists a scalar function  $\lambda = \lambda(\mathbf{x})$  such that

$$\nabla \times \mathbf{B} = \lambda \mathbf{B}. \quad (1.2)$$

Within the general class of linear force-free equilibria described by (1.2), Taylor states or Woltjer–Taylor states are particular equilibria for which  $\lambda$  is a spatially uniform constant given by the ratio of the magnetic energy to the magnetic helicity, see Chapter 11 of [6]. They play a central role in plasma physics [5,11,15,21,22,40,54,59] as the natural state resulting from

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dissipative turbulent relaxation [50,55]. Since they satisfy the equation  $\nabla \times \mathbf{B} = \lambda \mathbf{B}$  with  $\lambda$  constant, magnetic fields in Taylor state configurations are a special case of a class of force-free fields called linear Beltrami fields [3,7]. Since in this work we will consider  $\lambda$  to be a given input to the solver, we will use the expressions *Taylor state* and *linear Beltrami field* interchangeably for the remainder of this article.

Linear Beltrami fields have been extensively studied mathematically, and their properties are well understood by now [19, 25,39,43,49]. On the other hand, relatively few numerical solvers have been developed to compute them in geometries relevant to plasma physics. To the best of the authors' knowledge, solvers for these problems based on integral equation formulations have never been constructed despite their desirable properties: access to relatively high-precision derivatives of the field (via analytic differentiation of the integral representation), low memory requirements (only the boundary has to be discretized), and overall rapid convergence of the solution (when coupled with high-order quadrature rules and a fast algorithm, such as a fast multipole method).

We take a moment to justify this claim. One may at first think that Taylor states in axisymmetric toroidal geometries can be viewed as a special class of more general Grad-Shafranov equilibria [32,52], as was for instance done in [14]. From this point of view, Grad-Shafranov solvers relying on integral formulations [44,48] could be used to compute linear Beltrami fields. However, this approach is not satisfactory for the following reasons. First, a Grad-Shafranov solver would not take advantage of the particular properties of linear Beltrami fields. Second, certain applications [11,40] require the computation of linear Beltrami fields in hollow toroidal shells. Grad-Shafranov solvers are usually not designed to handle such geometries. Finally, and most importantly, Taylor states in axisymmetric domains may not be axisymmetric themselves [55]. By definition, the Grad-Shafranov equation does not apply to these fully three-dimensional, bifurcated states.

The purpose of this article is to present the first integral equation solver for the calculation of Taylor states in toroidal regions. While preliminary results from this work were given in [25], details of the actual solver were not provided. A separation of variables numerical solver for the full *exterior* axisymmetric electromagnetic scattering problem from perfect conductors is discussed in [26], but this work does not address the computation of interior eigenfunctions nor solve the boundary value problem with data on the normal components of  $\mathbf{E}$ ,  $\mathbf{H}$ . We close this gap with the present work. The integral formulation we present here applies to both toroidally axisymmetric and non-axisymmetric domains, but thus far our numerical solver can only treat the first situation. We will therefore restrict the description of the numerical solver to that case. Let us stress again that while the domain is axisymmetric, the Taylor state itself may not be, and the solver we present here can compute these non-axisymmetric equilibria. As such, it may be applied to the computation of magnetohydrodynamic equilibria in spheromaks [29], reversed field pinches [56], and in tokamaks for start-up scenarios [5] and the study of magnetohydrodynamic instabilities [31,34].

The mathematically well-posed form of the problem is as follows. We construct numerical solutions to the Beltrami boundary-value problem given by:

$$\begin{aligned} \nabla \times \mathbf{B} &= \lambda \mathbf{B} & \text{in } \Omega, \\ \mathbf{B} \cdot \mathbf{n} &= 0 & \text{on } \Gamma, \end{aligned} \tag{1.3}$$

where  $\lambda$  is a real number given as input to the solver,  $\Omega$  is an axisymmetric toroidal domain, and  $\Gamma = \partial\Omega$  is the (smooth) boundary of the region  $\Omega$ . Depending on the genus of  $\Gamma$ , additional (topological) constraints on  $\mathbf{B}$  must be added in order for (1.3) to be well-posed. For Taylor states in laboratory plasmas, it is often natural to take these constraints as conditions on the toroidal and poloidal flux of  $\mathbf{B}$  [40,55], see Fig. 1(a). In a genus-two toroidal flux shell (see Fig. 1(b)) two conditions must be imposed:

$$\int_{S_t} \mathbf{B} \cdot d\mathbf{a} = \Phi^{\text{tor}} \quad \text{and} \quad \int_{S_p} \mathbf{B} \cdot d\mathbf{a} = \Phi^{\text{pol}}, \tag{1.4}$$

where  $d\mathbf{a} = \mathbf{n} da$ , with  $da$  being the surface area element and  $\mathbf{n}$  the oriented normal along the surfaces  $S_t$  and  $S_p$ . On the other hand, if the toroidal domain is not hollow (genus-one), only one additional flux condition is necessary. The need for extra conditions (1.4) to ensure well-posedness stems from the multiply-connectedness of the boundary  $\Gamma$  – namely, the existence of *harmonic surface vector fields* on  $\Gamma$  and interior volume  $\lambda$ -Neumann vector fields in  $\Omega$  [23]. Readers interested in more details on the well-posedness of the boundary value problem ((1.3), (1.4)) may read references [25,43].

Our integral equation formulation is based on the observation that if  $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ , then the pair  $\{\mathbf{E}, \mathbf{H}\} = \{i\mathbf{B}, \mathbf{B}\}$  satisfies the time-harmonic Maxwell's equations in vacuum, with  $\lambda$  playing the role of a wavenumber. Boundary conditions on the normal component of  $\mathbf{B}$  then correspond to boundary conditions on the normal components of  $\mathbf{E}$  and  $\mathbf{H}$ . This fact, coupled with the symmetry of  $\mathbf{E}$  and  $\mathbf{H}$ , makes it natural to represent  $\mathbf{B}$  using generalized Debye sources, as in [23,25]. Application of the boundary condition in (1.3) and flux constraints (1.4) to the generalized Debye source representation immediately yields a second-kind integral equation which can be solved with standard techniques.

The paper is organized as follows. In Section 2, we establish the link between linear Beltrami fields and the generalized Debye representation at the heart of our integral equation formulation. In Sections 3 and 4, we derive the second-kind integral equations for the densities of the vector and scalar potentials in the generalized Debye representation of the Beltrami field. Section 3 applies to toroidal regions, while Section 4 applies to hollow toroidal shells. Section 5 describes our numerical method to compute the solution to the integral equations, and to subsequently evaluate the Beltrami fields. In

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