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The lowest-order weak Galerkin finite element method for the Darcy equation on quadrilateral and hybrid meshes

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ABSTRACT

This paper investigates the lowest-order weak Galerkin finite element method for solving the Darcy equation on quadrilateral and hybrid meshes consisting of quadrilaterals and triangles. In this approach, the pressure is approximated by constants in element interiors and on edges. The discrete weak gradients of these constant basis functions are specified in local Raviart–Thomas spaces, specifically RT_0 for triangles and unmapped $RT_{[0]}$ for quadrilaterals. These discrete weak gradients are used to approximate the classical gradient when solving the Darcy equation. The method produces continuous normal fluxes and is locally mass-conservative, regardless of mesh quality, and has optimal order convergence in pressure, velocity, and normal flux, when the quadrilaterals are asymptotically parallelograms. Implementation is straightforward and results in symmetric positive-definite discrete linear systems. We present numerical experiments and comparisons with other existing methods.

1. Introduction

This paper concerns finite element methods for the Darcy equation

$$\begin{cases} \nabla\cdot(-\mathbf{K}\nabla p) \equiv \nabla\cdot\mathbf{u} = f, & \mathbf{x}\in\Omega, \\ p = p_D, & \mathbf{x}\in\Gamma^D, \\ \mathbf{u}\cdot\mathbf{n} = u_N, & \mathbf{x}\in\Gamma^N, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded (polygonal) domain, p the unknown pressure, $\mathbf{K}$ a permeability matrix that is uniformly symmetric positive-definite, f a source term, p_D and u_N are Dirichlet and Neumann boundary data respectively, and $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$, which has a nonoverlapping decomposition $\Gamma^D\cup\Gamma^N$.

Efficient and robust solvers for the Darcy equation are fundamentally important for numerical simulations of flow and transport in porous media. Some early work can be found in [15,16,31]. The two most important properties desired for Darcy solvers are local mass-conservation and normal flux continuity. For the continuous Galerkin (CG) methods, these can be achieved by postprocessing [14]. Discontinuous Galerkin methods are locally mass-conservative by design, and postprocessing will provide normal flux continuity [7]. Mixed finite element methods (MFEMs) have both properties by design [8], but indefinite linear systems need to be solved. The enriched Galerkin methods attain these properties by enriching the CG approximation spaces with discontinuous piecewise constants [32]. Other research efforts can be found in [9].

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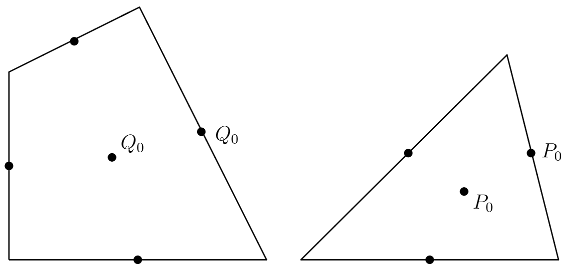

Fig. 1. *Left*: WG(Q_0, Q_0; $RT_{[0]}$) element on a quadrilateral; *Right*: WG(P_0, P_0; RT_0) element on a triangle.

The development of Darcy solvers on quadrilateral meshes is more challenging than those for rectangular and triangular meshes. In [4,5,36], the Piola transform is utilized to maintain normal flux continuity on general quadrilateral meshes.

The recently developed weak Galerkin (WG) finite element methods [33] bring a new perspective. When applied to the Darcy equation on triangular and rectangular meshes [23,24], the WG methods satisfy the aforementioned two physical properties, have optimal order convergence rates, and result in symmetric positive-definite discrete linear systems. Under certain conditions [21], the WG methods satisfy the discrete maximum principle. However, Darcy solvers on rectangular meshes have limited use for real applications. In certain cases, quadrilateral meshes or hybrid meshes consisting of quadrilaterals and triangles are desired to accommodate complicated domain geometry while keeping degrees of freedom low [1,10,39]. WG methods have been developed for elliptic equations on general polygonal meshes [29,28] and these methods can be applied to solve the Darcy equation on quadrilateral and triangular meshes. However, existing WG methods require a stabilization parameter, for which an optimal value is generally unknown. Further, although these WG methods produce a continuous normal flux, it is unclear how a numerical velocity can be calculated.

In this paper, we develop a new WG method for solving the Darcy equation on quadrilateral and hybrid meshes. This new method uses the lowest order approximation for pressure, namely, constants inside elements and on edges, is locally conservative, and produces continuous normal fluxes regardless of mesh quality. When the quadrilaterals are asymptotically parallelograms [5], this method has optimal order convergence in pressure, velocity, and flux.

The rest of this paper is organized as follows. In Section 2, we discuss the construction of WG(Q_0, Q_0; $RT_{[0]}$) finite elements on quadrilaterals and WG(P_0, P_0; RT_0) on triangles. Section 3 presents the Darcy solver using the lowest order WG finite elements, namely WG(Q_0, Q_0; $RT_{[0]}$) for quadrilaterals and WG(P_0, P_0; RT_0) for triangles. The properties of this Darcy solver are discussed in Section 4 and error estimates are developed. Section 5 briefly discusses two other finite element methods developed for Darcy problems, a mixed method that uses the Piola transform [3] and a WG method with stabilization [28]. Section 6 presents numerical experiments using the new WG method developed here and compares them with results obtained using a method incorporating the Piola transform and an existing WG method. Section 7 concludes the paper with some remarks.

2. Lowest-order WG elements on quadrilaterals and triangles

WG(P_0, P_0; RT_0) finite elements on triangles and WG(Q_0, Q_0; $RT_{[0]}$) finite elements on rectangles are used for solving the Darcy equation in [23], but in this section we construct WG(Q_0, Q_0; $RT_{[0]}$) finite elements for quadrilaterals. (See Fig. 1.) These will include the rectangular WG(Q_0, Q_0; $RT_{[0]}$) elements as a special case. This WG approach is different than the approach used in [5], which employs the Piola transform to construct mixed finite elements. The mixed finite element methods enforce normal flux continuity at the level of finite element spaces. The WG approach in this paper uses a larger finite element space for gradients, but normal flux continuity is attained through the bilinear form and the properties of discrete weak gradients.

2.1. WG(Q_0, Q_0; $RT_{[0]}$) finite elements on quadrilaterals

Let E be a convex quadrilateral with center (x_c, y_c). Let $X = x - x_c$, $Y = y - y_c$. We denote

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} X \\ 0 \end{bmatrix}, \quad \mathbf{w}_4 = \begin{bmatrix} 0 \\ Y \end{bmatrix}, \tag{2}$$

and define a local Raviart–Thomas space as

$$RT_{[0]}(E) = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4). \tag{3}$$

The Gram matrix of the above basis is obviously