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Spatial eigensolution analysis of energy-stable flux reconstruction schemes and influence of the numerical flux on accuracy and robustness

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ABSTRACT

This study focuses on the dispersion and diffusion characteristics of high-order energy-stable flux reconstruction (ESFR) schemes via the spatial eigensolution analysis framework proposed in [1]. The analysis is performed for five ESFR schemes, where the parameter '*c*' dictating the properties of the specific scheme recovered is chosen such that it spans the entire class of ESFR methods, also referred to as VCJH schemes, proposed in [2]. In particular, we used five values of '*c*', two that correspond to its lower and upper bounds and the others that identify three schemes that are linked to common high-order methods, namely the ESFR recovering two versions of discontinuous Galerkin methods and one recovering the spectral difference scheme. The performance of each scheme is assessed when using different numerical intercell fluxes (e.g. different levels of upwinding), ranging from "under-" to "over-upwinding". In contrast to the more common temporal analysis, the spatial eigensolution analysis framework adopted here allows one to grasp crucial insights into the diffusion and dispersion properties of FR schemes for problems involving non-periodic boundary conditions, typically found in open-flow problems, including turbulence, unsteady aerodynamics and aeroacoustics.

1. Introduction

Computational fluid dynamics (CFD) is facing the challenge to expand its current capabilities to flow problems that can be only marginally described by the prevailing numerical methodologies adopted today [3]. The automotive and aerospace industries, for instance, have been using low-order numerical techniques in conjunction with approximated and steady-state-tailored approximation strategies, such as Reynolds-Averaged Navier–Stokes (RANS) approaches or Detached-Eddy simulation (DES), for many years and their use is still almost ubiquitous. While these numerical technologies are numerically robust and the associated engineering workflow is well-established, they struggle to accurately describe a wide range of problems that are of practical interest in various branches of engineering and applied sciences and that involve unsteady turbulent flows at high Reynolds numbers. The ability of accurately predict the behavior of the latter class of problems is particularly

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relevant for expanding the capabilities of computer-aided computational fluid dynamics (CFD) to off-design cycle conditions and for possibly reducing the number of (commonly expensive) wind-tunnel tests required for a given aerodynamic configuration.

It is therefore essential to explore alternative ways to enhance the predictive skills of CFD and facilitate their adoption in the broader industrial community working in the field [3,4]. From this perspective, high-fidelity computations relying on high-order numerical methods, namely spectral element methods including discontinuous Galerkin and flux reconstruction approaches [5–10], are particularly attractive, especially when used for large-eddy simulation (LES) and under-resolved direct numerical simulation (uDNS) of high Reynolds number turbulent flows — e.g. [11–20]. They in fact offer better (tunable) dispersion and diffusion properties over traditional low-order schemes, an aspect that constitute the key for properly describing turbulent flows, and have a superior resolution power per degree of freedom [21,22]. In addition, they are particularly suited to discretize complex geometries, due to the possibility of using high-order meshing strategies for curved surfaces, aspect that is particularly crucial in many industry-relevant problems.

However, in order to achieve numerically robust simulations and accurate data for under-resolved flow computations, it is of fundamental importance to better understand the numerical characteristics of the underlying numerics regarding wave-like solution components. The present study investigates the dispersion and diffusion/dissipation properties of energy-stable flux reconstruction (i.e. ESFR) spectral element methods (also referred to as VCJH schemes) [7,2]. The analysis is carried out by means of the spatial eigensolution analysis framework applied to a one-dimensional linear advection problem that was initially proposed in [1] for the discontinuous Galerkin method. In order to investigate different VCJH methods, we vary the parameter 'c' that controls the particular VCJH scheme recovered [2]. Whilst the spatial eigensolution analysis framework and can be applied to the entire class of flux reconstruction (FR) methods — not necessarily VCJH schemes — we focus on five VCJH schemes that are most commonly found in the literature. These encompass the FR schemes recovering a nodal discontinuous Galerkin method, $\mathrm{FR_{DG}}$, the spectral difference scheme, $\mathrm{FR_{SD}}$, and the Huynh or g2 scheme, $\mathrm{FR_{HU}}$. In addition, we take into account two schemes that are at the lower and upper bounds of the scalar parameter 'c', that are $\mathrm{FR}_{c_{-}/2}$ (lower bound) and FR_{c_∞} (upper bound, $c \to \infty$). The study presented here includes how different intercell numerical fluxes affect the diffusion and dispersion properties of the FR schemes considered. This investigation is performed as part of the spatial eigensolution analysis, where we use a scalar parameter β in the definition of the numerical flux to control the amount of upwinding allowed at the interfaces between the various elements of the given FR spatial discretization considered. The latter study is particularly relevant in the context, for instance, of compressible flow simulations, where the use of more simplistic (e.g. local Lax–Friedrichs or Rusanov and HLL) vs. more complete (e.g. HLLC, Roe) Riemann solvers can severely affect the accuracy of the simulation and its numerical robustness. The study proposed in this paper provides crucial insights into the numerical characteristics, namely diffusion and dispersion, of ESFR schemes for open flow problems subject to generic inflow/outflow boundary conditions, that are frequently found in several CFD applications, and provides essential guidelines for the best practices to adopt for the numerical intercell fluxes (e.g. Riemann solvers for compressible flows). The findings from the spatial eigensolution analysis are successively confirmed by a one-dimensional linear advection test case and by a two-dimensional test case that resembles the behavior of a spatially evolving turbulent flow at very high Reynolds number, where the mesh adopted is necessarily under-resolved.

Eigensolution analysis has been already applied to spectral element methods — e.g. [23–26,21,27,28] and to the FR method in particular [29]. Nevertheless, most of the dedicated literature is related to the temporal approach, that is relevant to periodic problems and does not investigate the effects of the intercell numerical fluxes, that are shown here to be of fundamental importance when dealing with under-resolved computations of high-Reynolds number turbulent flows. In summary, this work applies the spatial eigensolution framework proposed in [1] to the FR approach and highlights the diffusion and dispersion properties of energy-stable FR schemes when using different numerical fluxes — e.g. Riemann solvers — for under-resolved flow simulations relevant to real-world problems.

This paper is organized as follows. Section 2 introduces the FR approach. Section 3 presents the spatial eigensolution analysis framework for the FR method and addresses the results. Section 4 outlines the results for the one-dimensional linear advection equation that is used to verify the results of the spatial eigensolution analysis. Section 5 shows a two-dimensional test case that is proposed to verify the insights obtained from the eigenanalysis in a more complex scenario. Section 6 highlights the main findings and conclusions of the paper.

2. The flux reconstruction approach

In this section, we introduce the flux reconstruction (FR) approach for a generic one-dimensional scalar conservation law, section 2.1, and we outline the ESFR (equivalently VCJH) class of schemes investigated here, section 2.2.

2.1. One-dimensional scalar conservation law

Consider the one-dimensional scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0\,, \tag{1}$$