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A stable and high-order accurate discontinuous Galerkin based splitting method for the incompressible Navier–Stokes equations

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ABSTRACT

In this paper we consider discontinuous Galerkin (DG) methods for the incompressible Navier–Stokes equations in the framework of projection methods. In particular we employ symmetric interior penalty DG methods within the second-order rotational incremental pressure correction scheme. The major focus of the paper is threefold: i) We propose a modified upwind scheme based on the Vijayasundaram numerical flux that has favourable properties in the context of DG. ii) We present a novel postprocessing technique in the Helmholtz projection step based on $H(\mathrm{div})$ reconstruction of the pressure correction that is computed locally, is a projection in the discrete setting and ensures that the projected velocity satisfies the discrete continuity equation exactly. As a consequence it also provides local mass conservation of the projected velocity. iii) Numerical results demonstrate the properties of the scheme for different polynomial degrees applied to two-dimensional problems with known solution as well as large-scale three-dimensional problems. In particular we address second-order convergence in time of the splitting scheme as well as its long-time stability.

1. Introduction

The application of discontinuous Galerkin (DG) methods to the Navier–Stokes equations is popular due to their potentially high order of convergence, the inf-sup stability and local mass conservation property [1–3]. The latter is generally not fulfilled for conforming finite element discretizations. In addition to the 2×2 block structure arising from the saddle point system discontinuous Galerkin methods offer a further block structure when the unknowns associated with one cell of the mesh are grouped together. This data structure is essential for high-performance implementations of the discontinuous Galerkin method [4,5] as it avoids costly memory gather and scatter operations when compared to conforming finite element methods.

Operator splitting methods for solving the instationary Navier–Stokes equations has been subject to detailed investigations for the recent decades. One possibility in the splitting methods is to split between the convective term and the saddle point structure which is realized in Glowinski's Θ-scheme, [6,7]. Another possibility is to split between incompressibility and dynamics which has been independently developed by Chorin [8] and Témam [9] and is referred to as Chorin's projec-

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tion method. The latter splitting schemes have the appealing feature that at each time step, instead of solving a saddle point system, one only has to solve a vector-valued heat equation for the velocity (in the Stokes case) and a Poisson equation for the pressure. The choice of artificial boundary conditions on the pressure Poisson equation is a delicate issue in projection methods of this class [10–12]. Several higher-order extensions of Chorin's first order method have been suggested in the literature [13–18]. Here we concentrate on the classic incremental pressure-correction scheme (IPCS) [19] and the rotational incremental pressure-correction scheme (RIPCS) [14].

The use of a DG spatial discretization within splitting schemes is a current subject of active research. A naive computation of the divergence free velocity by subtraction of the rotation free part is reported to be unstable when the spatial mesh is coarse and the time step is small, see [20,21,4], where several local postprocessing techniques are discussed to overcome this difficulty. In this paper we propose a new postprocessing technique based on $H(\mathrm{div})$ reconstruction of the discrete pressure gradient which is popular in porous media flow computations [22,23]. The new approach provides a discrete velocity that satisfies the discrete continuity equation exactly and in consequence is locally mass conservative and defines a projection. These properties are not satisfied by the postprocessing schemes available in the literature.

The structure of the paper is organized as follows: In section 2 we recapitulate the discontinuous Galerkin discretization by the interior penalty method as presented in [2,1]. In section 3 we discuss the Helmholtz decomposition, prove our main result and present the projection methods In section 4 we elaborate on numerical experiments for the discontinuous Galerkin discretization based on the reference problems by [15,16,24,18] and assess the properties of the new postprocessing scheme.

2. Discontinuous Galerkin discretization of the incompressible Navier–Stokes equations

In this section we present the spatial discretization of the Navier–Stokes system with an interior penalty DG method taken from [2]. The convective term is discretized using the Vijayasundaram flux.

The instationary incompressible Navier–Stokes equations in an open and bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ determining the velocity v and pressure p for a right-hand side f, constant viscosity μ and density ρ are given by

$$\rho\partial_t v - \mu\Delta v + \rho(v\cdot\nabla)v + \nabla p = f \qquad \text{in } \Omega \times (0, T] \tag{1a}$$

$$\nabla\cdot v = 0 \qquad \text{in } \Omega\times(0,T] \tag{1b}$$

$$v = v_0 \qquad \text{for } t = 0 \tag{1c}$$

and either Dirichlet boundary condition for the velocity:

$$v = g \qquad \text{on } \Gamma_D = \partial\Omega,\, t\in(0,T] \tag{1d}$$

together with

$$\int_\Omega p\,\mathrm{d}x = 0 \qquad \text{for all } t\in(0,T] \tag{1e}$$

or mixed boundary conditions:

$$v = g \qquad \text{on } \Gamma_D \neq \{\partial\Omega, \emptyset\} \tag{1f}$$

$$\mu\nabla v n - pn = 0 \qquad \text{on } \Gamma_N = \partial\Omega\setminus\Gamma_D \tag{1g}$$

with $(0, T]$ being the time interval of interest. For pure Dirichlet boundary conditions g is required to satisfy the compatibility condition $\int_{\partial\Omega} g\cdot n\mathrm{d}x = 0$. In the numerical examples below we will also consider periodic boundary conditions in addition. Under appropriate assumptions the Navier–Stokes problem in weak form has a solution (v, p) in $(H^1(\Omega))^d \times L^2(\Omega)$ for $t\in(0,T]$, [25,26]. In case of pure Dirichlet boundary conditions the pressure is only determined up to a constant and is in the space $L^2_0(\Omega) = \{q\in L^2(\Omega) \mid \int_\Omega q\mathrm{d}x = 0\}$.

For the discretization let $\mathcal{E}_h$ be an affine cubic mesh (the restriction to affine meshes is only needed when the Raviart–Thomas reconstruction is used) with maximum diameter h. We denote by Γ_h^{int} the set of all interior faces, by Γ_h^D the set of all faces intersecting with the Dirichlet boundary Γ_D and by Γ_h^N the set of all faces intersecting with the mixed boundary Γ_N. We set $\Gamma_h = \Gamma_h^{\mathrm{int}} \cup \Gamma_h^D \cup \Gamma_h^N$. To an interior face $e\in\Gamma_h^{\mathrm{int}}$ shared by elements E_e^1 and E_e^2 we define an orientation through its unit normal vector n_e pointing from E_e^1 to E_e^2. The jump and average of a scalar-valued function ϕ on a face is then defined by

$$\begin{aligned}
[\phi] &= \phi\,|_{E_e^1} - \phi\,|_{E_e^2} = \phi^{\mathrm{int}} - \phi^{\mathrm{ext}}, \\
\{\phi\} &= \frac{1}{2}\phi\,|_{E_e^1} + \frac{1}{2}\phi\,|_{E_e^2} = \frac{1}{2}\phi^{\mathrm{int}} + \frac{1}{2}\phi^{\mathrm{ext}}.
\end{aligned} \tag{2}$$