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A fast direct solver for boundary value problems on locally perturbed geometries

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ABSTRACT

Many applications including optimal design and adaptive discretization techniques involve solving several boundary value problems on geometries that are local perturbations of an original geometry. This manuscript presents a fast direct solver for boundary value problems that are recast as boundary integral equations. The idea is to write the discretized boundary integral equation on a new geometry as a low rank update to the discretized problem on the original geometry. Using the Sherman–Morrison formula, the inverse can be expressed in terms of the inverse of the original system applied to the low rank factors and the right hand side. Numerical results illustrate for problems where perturbation is localized the fast direct solver is three times faster than building a new solver from scratch.

1. Introduction

This manuscript presents a fast direct solver for boundary integral equations where the geometry for each problem corresponds to a local perturbation of the original geometry. In particular, we are interested in problems where the local perturbation to the geometry is much smaller than the original geometry. Since a direct solver is constructed, the technique is ideal for problems with many right hand sides and/or suffer from ill-conditioning due to geometric complexity. Boundary value problems involving locally perturbed geometries arise in a variety of applications such as optimal design [24], and adaptive discretization techniques [6]. For example, finding the optimal placement of an attachment to a large geometry which minimizes the radar cross section involves solving many problems where the local perturbation is the same but the placement on the boundary changes.

For many boundary integral equations, the linear system that results from the discretization of an integral equation is amenable to fast direct solvers such as those built from hierarchically semiseparable (HSS) [25,5,20], $\mathcal{H}$-matrix [13], hierarchically block separable (HBS) [8], hierarchical interpolatory factorization (HIF) [16] and hierarchical off-diagonal low rank (HODLR) [1] representations. These direct solvers utilize the fact that the off-diagonal blocks of the dense matrix are low rank. The different variants correspond to different ways of exploiting this property. Let $\mathbf{A}$ denote the matrix resulting from discretization of the boundary integral equation. The factored approximation of the matrix $\mathbf{A}$, denoted by $\tilde{\mathbf{A}}$, is constructed so that $\|\mathbf{A}-\tilde{\mathbf{A}}\| \leq \epsilon$ for a user defined tolerance ϵ. $\tilde{\mathbf{A}}$ is called the *compressed representation* of $\mathbf{A}$. The inverse of the compressed matrix is then constructed via a variant of a Woodbury formula or by expanding the matrix out to a larger sparse system and using a sparse direct solver. We refer the reader to the references for further details.

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Building from the approach in [10], the solution technique presented in this paper casts the linear system for problem on the perturbed geometry as an extended linear system which consists of a two-by-two block diagonal matrix plus a low rank update. The block diagonal matrix has a block equal to the matrix for the original geometry. By using the Sherman–Morrison formula, the approximate inverse of the original system can be exploited and the approximate inverse of the extended system can be applied rapidly. The compressed representation matrix in the original system is utilized to reduce the cost of computing the low rank factorization of the update matrix. The techniques used in this work draw from earlier work in [21,3,22,26].

The method presented in this paper is ideally suited for problems where the local perturbation is the same over many placements on the boundary or the number of removed points on the boundary is not large. Since the solution technique does not modify the original compressed representation, it can be combined with any fast direct solver.

1.1. Related work

The paper [23] presents a technique for updating the *Hierarchical interpolative factorization* (HIF) of the matrix **A**. This task involves locating and updating the relevant low rank factors and (potentially) modifying the underlying hierarchical tree structure. An approximate inverse is then constructed for the updated compressed representation. The inversion step is one of the most expensive steps in the precomputation of a fast direct solver.

The method in [17] is designed for point-wise perturbations in the geometry. Like the method presented in this paper, [17] writes the change as a low rank update to **A** and utilizes the Sherman–Morrison formula. The factors are taken so that one is a submatrix of the identity matrix and the other is a dense matrix corresponding to the subtraction and addition of points. Additional compression or utilization of fast direct solver information for the solver on the original geometry is not utilized. Instead dense matrix–matrix multiplication is precomputed so that application of the Sherman–Morrison formula requires pulling the appropriate submatrices. In contrast, the method in [18] creates new fast direct solver from scratch for each new geometry thus making it able to handle more general geometries.

Unlike the method in [17], the new direct solver is able to handle more general changes and utilizes as much precomputation as possible from the solver on the original geometry. The solver in [23] utilizes some of the factorizations from the original solver but it is not able to make use of the already computed inverse of the original system. [18] does not utilize information from the original system.

1.2. Outline of paper

This manuscript begins by presenting the boundary integral formulation, the discretized linear system and the extended linear system for a model problem in section 2. The fast direct solver presented in this paper for a locally perturbed geometry utilizes the factors computed in the fast direct solver for the original geometry. While the method can be utilized in conjunction with any fast direct solver, we chose to review the HBS method and its physical interpretation in section 3 for simplicity of presentation. Next, the construction of the new fast direct solver is presented in section 4. Finally numerical experiments report on the performance of the solver in section 5. Section 6 reviews the methods and highlights the potential of the solution technique.

2. Model problem

This section begins by reviewing the boundary integral approach for solving a Laplace boundary value problem. Then the technique for writing the linear system corresponding to the discretized boundary value problem on a geometry that is a local perturbation of the original is presented in section 2.2.

2.1. Boundary integral equations

For simplicity of presentation, we consider the Laplace boundary value problem

$$\begin{aligned} -\Delta u(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad \mathbf{x} \in \Gamma. \end{aligned} \tag{1}$$

Fig. 1(a) illustrates a sample geometry. The vector $\boldsymbol{\nu}_{\mathbf{y}}$ denotes the outward normal vector at the point $\mathbf{y} \in \Gamma$. For $\mathbf{x} \in \Omega$, we represent the solution to (1) as a double layer potential

$$u(\mathbf{x}) = \int_{\Gamma} D(\mathbf{x}, \mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}), \qquad \mathbf{x} \in \Omega, \tag{2}$$

where $D(\mathbf{x}, \mathbf{y}) = \partial_{\boldsymbol{\nu}_{\mathbf{y}}} G(\mathbf{x}, \mathbf{y})$ is double layer kernel, $G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi}\log|\mathbf{x} - \mathbf{y}|$ is the fundamental solution and $\sigma(\mathbf{x})$ is the unknown boundary charge distribution. By taking the limit of $u(\mathbf{x})$ as $\mathbf{x}$ goes to the boundary and setting it equal to $g(\mathbf{x})$, we find the boundary charge distribution $\sigma(\mathbf{x})$ satisfies the following boundary integral equation