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A modal approach based on perfectly matched layers for the forced response of elastic open waveguides

M. Gallezot, F. Treyssède*, L. Laguerre

IFSTTAR, GERS, GeoEND, F-44344, Bouguenais, France

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ABSTRACT

This paper investigates the computation of the forced response of elastic open waveguides with a numerical modal approach based on perfectly matched layers (PML). With a PML of infinite thickness, the solution can theoretically be expanded as a discrete sum of trapped modes, a discrete sum of leaky modes and a continuous sum of radiation modes related to the PML branch cuts. Yet with numerical methods (*e.g.* finite elements), the waveguide cross-section is discretized and the PML must be truncated to a finite thickness. This truncation transforms the continuous sum into a discrete set of PML modes. To guarantee the uniqueness of the numerical solution of the forced response problem, an orthogonality relationship is proposed. This relationship is applicable to any type of modes (trapped, leaky and PML modes) and hence allows the numerical solution to be expanded on a discrete sum in a convenient manner. This also leads to an expression for the modal excitability valid for leaky modes. The physical relevance of each type of mode for the solution is clarified through two numerical test cases, a homogeneous medium and a circular bar waveguide example, excited by a point source. The former is favourably compared to a transient analytical solution, showing that PML modes reassemble the bulk wave contribution in a homogeneous medium. The latter shows that the PML mode contribution yields the long-term diffraction phenomenon whereas the leaky mode contribution prevails closer to the source. The leaky mode contribution is shown to remain accurate even with a relatively small PML thickness, hence reducing the computational cost. This is of particular interest for solving three-dimensional waveguide problems, involving two-dimensional cross-sections of arbitrary shapes. Such a problem is handled in a third numerical example by considering a buried square bar.

1. Introduction

1.1. Context and state-of-the-art

Elastic guided waves are interesting for many applications involving elongated structures (*e.g.* non-destructive evaluation (NDE), structural health monitoring (SHM), exploration geophysics...), because of their ability to propagate over large distances. When the structure (the core) is embedded into a large solid matrix, it can be considered as an open waveguide (unbounded in the transverse direction). Such a configuration typically occurs in civil engineering and in geophysics for

* Corresponding author.
E-mail addresses: matthieu.gallezot@ifsttar.fr (M. Gallezot), fabien.treyssede@ifsttar.fr (F. Treyssède), laurent.laguerre@ifsttar.fr (L. Laguerre).

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example. Contrary to waveguides in vacuum (closed waveguides), most of the waves in open waveguides are attenuated by leakage in the surrounding medium as they propagate. Moreover, the underlying physics is deeply transformed.

Indeed, in open waveguides, three main kinds of modes are distinguished: trapped modes, radiation modes, and leaky modes. Trapped modes propagate without leakage attenuation along the waveguide axis and decay exponentially in the transverse direction. These waves are confined in the core of the waveguide or at the interface. Their existence depends on material contrasts between the core and the surrounding medium [1]. Radiation modes are standing waves in the transverse direction, which are propagative or evanescent in the axis direction [2,3]. Finally, leaky modes propagate with leakage attenuation along the waveguide axis. These modes dramatically grow exponentially in the transverse direction [2,4,5].

Modes are computed by considering the source-free problem. This can be done by analytical methods (*e.g.* the Thomson–Haskell [6,7], the stiffness matrix [8] or the global matrix [9] methods), which are yet limited to canonical waveguide geometries. Numerical methods are well-suited for more complex geometries. The idea is then to discretize only the cross-section of the waveguide while describing analytically the direction of the wave propagation. In closed waveguides, this approach yields a linear eigenvalue problem. It has been applied under various names in the literature, such as the extended Ritz technique [10,11]; the thin layer method (TLM) for stratified waveguides in geophysics [12,13]; the strip-element method [14], the Semi-Analytical Finite Element method (SAFE) [15,16] or more recently the Scaled Boundary Finite Element Method (SBFEM) [17,18] in ultrasonics. In this paper, this approach will be referred to as *waveguide formulation* to avoid the use of acronyms.

Extending the waveguide formulation to open waveguides is not straightforward because of the unbounded nature of the problem in the transverse direction. This difficulty is enhanced by the transverse growth of leaky modes. Therefore, the waveguide formulation must be coupled to other techniques to numerically compute the modes of open waveguides.

The first class of methods avoids the discretization of the embedding medium by using appropriate boundary conditions. The waveguide formulation has been combined with the boundary element method to model three-dimensional waveguides immersed in fluids [19,20] or embedded in solids [21]. Similarly, exact boundary conditions have been proposed for two-dimensional waveguides (plates and cylinders) immersed in fluids [22]. All of these boundary conditions lead to a highly non-linear eigenproblem that is difficult to solve. The latter can be linearized in the case of two-dimensional plates immersed in perfect fluids [23,24]. In the case of high-contrast solid waveguides, the waveguide formulation can also be coupled to an approximate condition (the so-called dashpot boundary condition) [25]. With this approximation, the eigenproblem remains linear.

The second class of methods requires a discretization of the surrounding medium, which must be truncated. The eigenproblem remains linear. To avoid spurious reflections due to truncation, the waveguide formulation have been combined with non-reflecting [26] and continued-fraction absorbing [27] boundary conditions in fluids, or paraxial approximation in solids [13,28]. Absorbing layers of artificially growing damping can also be used to simulate fluid [29] or solid [30] infinite media. Another technique consists in using a Perfectly Matched Layer (PML) to model the infinite surrounding medium (solid or fluid) [31–36]. Contrary to absorbing layers, the PML avoids most spurious reflections from the layer, which allows its thickness to be greatly reduced. Moreover, it have been shown that the computation of leaky modes with a PML is mathematically relevant (see Ref. [37] for scalar wave problems).

As far as the forced response problem is concerned, modal expansion methods have been widely applied in closed waveguides [38]. However, their application to open elastic waveguides is more intricate and has been barely considered in the literature. With a numerical approach, the particular case of two-dimensional plates immersed in fluids has recently been handled by a waveguide formulation with exact boundary conditions [24]. The case of a stratified plate over or between half-spaces has been treated using a PML in Ref. [36].

Theoretically, the forced response of an open waveguide can be expanded on trapped modes and radiation modes [2,3,39], such that the displacement field can be symbolically written as:

$$u(\mathbf{r},\omega) = \sum \text{trapped} + \int \text{radiation modes} \tag{1}$$

where, in addition to trapped modes, complex poles of backward type are also likely to occur depending on the problem type [40,41]. Let us briefly recall the origin of Eq. (1). In the wavenumber domain, the solution of the problem is a multivalued function owing its dependence on the transverse wavenumber of the unbounded medium. The transverse wavenumber is indeed the square root of a complex number on a two-sheeted Riemann surface. To evaluate analytically the inverse spatial Fourier transform of the solution, a branch cut is defined separating the proper Riemann sheet (where trapped modes occur) from the improper Riemann sheet (where leaky modes occur). Hence, the inverse transform integration is performed only on the proper sheet and gives rise [see Eq. (1)] to the discrete sum of trapped modes and to the continuum of radiation modes, which represents the branch cut contribution. This continuum is characteristic of the unboundedness of the modal problem and is difficult to manipulate from a mathematical point of view. In elastodynamics, there are two continua instead of one because two transverse wavenumbers occur (longitudinal and shear waves) [42,43]. The continua can actually be approximated with a convenient discrete set of leaky modes, *e.g.* using the steepest descent method [39,40]. This approximation is valid in a zone restricted near the core, in which leaky modes can provide useful practical information such as axial attenuation and travelling velocity of waves packets [44].

In a recent work [45], the authors have shown that when the surrounding medium is modified by an infinite PML, the forced response can theoretically be obtained with a modal expansion on trapped modes, revealed leaky modes and two