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A Hermite WENO reconstruction for fourth order temporal accurate schemes based on the GRP solver for hyperbolic conservation laws ☆

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ABSTRACT

This paper develops a new fifth order accurate Hermite WENO (HWENO) reconstruction method for hyperbolic conservation schemes in the framework of the two-stage fourth order accurate temporal discretization in Li and Du (2016) [13]. Instead of computing the first moment of the solution additionally in the conventional HWENO or DG approach, we can directly take the *interface values*, which are already available in the numerical flux construction using the generalized Riemann problem (GRP) solver, to approximate the first moment. The resulting scheme is fourth order temporal accurate by only invoking the HWENO reconstruction twice so that it becomes more compact. Numerical experiments show that such compactness makes significant impact on the resolution of nonlinear waves.

1. Introduction

In the development of high order accurate schemes for hyperbolic conservation laws, two families of approaches play important roles: one belongs to the method of line that achieves the temporal accuracy using the Runge–Kutta strategy [8,29,15,9,27]; the other is the Lax–Wendroff type approach that adopts the Cauchy–Kowalevski expansions to design temporal-spatial coupled schemes [11,1,3,17,22,33,20]. Either family of approaches have their own advantages and disadvantages. The former has the simplicity in their practical implementation thanks to exact or approximate Riemann solvers, but the multi-stage temporal iteration inevitably causes the enlargement of the size of stencils; the latter can avoid the multi-stage temporal iteration but have to repeatedly make the differentiation of governing equations in order to construct high order accurate numerical fluxes. A recent two-stage fourth order accurate temporal discretization based on the Lax–Wendroff type solvers [13,19] makes a compromise between these two families of methods: It just takes a two-stage iteration for the fourth order accuracy by using second order accurate temporal-spatial coupled Lax–Wendroff flow solvers so that half of reconstruction steps can be saved in comparison with the same accurate method and complicated successive differentiations of governing equations can be avoided, which could be further extended using the multi-derivative Runge–Kutta methods [19,6,35]. Moreover, we notice that the solution values on cell interfaces already available in the procedure of numerical

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flux construction, called *interface values* in the present paper, can be used for the reconstruction procedure, thanks to the Lax–Wendroff flow solvers, which motivates us for such a study.

We develop a new fifth order accurate Hermite WENO (HWENO) reconstruction in the framework of two-stage fourth order accurate temporal discretization [13]. The HWENO interpolation adopts two values: the average value of the solution and the corresponding averaged gradient value (the first moment), as usual. The novelty is that the gradient values are directly approximated using the interface values when the Lax–Wendroff type flow solvers are used [1,3,20,34], which is different from the standard HWENO method in [23,24,16]. Technically, we can further adjust nonlinear weights during the HWENO reconstruction, just like the WENO-Z method [4] modifying the classical WENO-JS [9]. In doing so, the resulting scheme is much more compact and has several distinct features.

(i) The scheme just uses half of the reconstruction steps, compared with the standard RK-WENO methods.
(ii) The interface values are already available in the computation of numerical fluxes and no extra efforts are made on the gradient approximation.
(iii) The interface values are approximated by using the GRP solver and thus they are strong solution values without taking account of possible discontinuities in trouble cells.
(iv) A single HWENO reconstruction is more compact than the standard WENO reconstruction [9], as shown in other HWENO schemes [23,24,16].

This paper is organized as follows. In Section 2, we quickly review the two-stage method based on the Lax–Wendroff flow solvers and the HWENO reconstruction methods. In Section 3, we show the gradient approximation over each computational cell by using interface values of solutions. In Section 4, several numerical examples are displayed for the performance of such a HWENO reconstruction, by comparing with the WENO reconstruction with the same numerical flux. A discussion is made in Section 5.

2. The two-stage fourth order method and the Hermite WENO reconstruction

This section serves to present a quick review of the two-stage fourth order method based on the Lax–Wendroff type flow solvers in [13] and the HWENO reconstruction procedure, originally in [23]. Instead of independently computing the first moment (the gradient of solution) in [23], we will construct it together with the solution average using the generalized Riemann problem (GRP) solver, which will be described in Section 3.

2.1. Review of the two-stage fourth-order scheme

The two-stage fourth-order finite volume schemes based on the GRP solver was developed in [13]. Certainly, we can also use Men'shov's modified GRP solver [17,18] and the ADER solver [33,34]. Both the acoustic and nonlinear versions of the GRP solver are provided in [3].

In this subsection, we quickly review this method by taking one-dimensional hyperbolic conservation laws,

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=0, \quad x\in\mathbb{R}, t>0,\\ &\mathbf{u}(x,0)=\mathbf{u}_0(x), \quad x\in\mathbb{R}, \end{aligned} \tag{2.1}$$

where $\mathbf{u}$ is a vector of conservative variables and $\mathbf{f}(\mathbf{u})$ is the associated flux function vector. Given the computational mesh $I_j=(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}})$ with the size $h=x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}$ for every j, we write (2.1) in form of the balance law,

$$\frac{d\bar{\mathbf{u}}_j(t)}{dt}=\mathcal{L}_j(\mathbf{u}):=-\frac{1}{h}[\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t))-\mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}},t))], \quad \bar{\mathbf{u}}_j(t)=\frac{1}{h}\int_{I_j}\mathbf{u}(x,t)dx, \tag{2.2}$$

where $\mathbf{u}(x_{j+\frac{1}{2}},t)$ is described in terms of GRP solver [3]. Then the two-stage approach for (2.1) is summarized as follows.

Step 1. With the cell averages $\bar{\mathbf{u}}_j^n$ and interface values $\hat{\mathbf{u}}_{j+\frac{1}{2}}^n$, reconstruct the data at t^n as a piece-wise polynomial function $\mathbf{u}(x,t^n)=\mathbf{u}^n(x)$ by the HWENO interpolation that will be described below, and compute the corresponding GRP value $(\mathbf{u}_{j+\frac{1}{2}}^n,(\partial\mathbf{u}/\partial t)_{j+\frac{1}{2}}^n)$.