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Discontinuous Skeletal Gradient Discretisation methods on polytopal meshes ☆

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A B S T R A C T

In this work we develop arbitrary-order Discontinuous Skeletal Gradient Discretisations (DSGD) on general polytopal meshes. Discontinuous Skeletal refers to the fact that the globally coupled unknowns are broken polynomials on the mesh skeleton. The key ingredient is a high-order gradient reconstruction composed of two terms: (i) a consistent contribution obtained mimicking an integration by parts formula inside each element and (ii) a stabilising term for which sufficient design conditions are provided. An example of stabilisation that satisfies the design conditions is proposed based on a local lifting of high-order residuals on a Raviart–Thomas–Nédélec subspace. We prove that the novel DSGDs satisfy coercivity, consistency, limit-conformity, and compactness requirements that ensure convergence for a variety of elliptic and parabolic problems. Links with Hybrid High-Order, non-conforming Mimetic Finite Difference and non-conforming Virtual Element methods are also studied. Numerical examples complete the exposition.

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1. Introduction

The numerical resolution of (linear or non-linear) partial differential equations (PDEs) is nowadays ubiquitous in the engineering practice. In this context, the design of convergent numerical schemes is a very active research topic. The Gradient Discretisation Method (GDM) is a recently introduced framework which identifies key design properties to obtain convergent schemes for a variety of linear and non-linear elliptic and parabolic problems. Several models of current use in fluid mechanics fall into the latter categories including, e.g., porous media flows governed by Darcy's law, phase change problems governed by the Stefan problem [36], as well as simplified models of the viscous terms in power-law fluids corresponding the Leray–Lions elliptic operators. The latter also appear in the modelling of glacier motion [38], of incompressible turbulent flows in porous media [26], and in airfoil design [37].

A Gradient Discretisation (GD) is defined by a finite-dimensional space encoding the discrete unknowns, as well as two linear operators acting on the latter, and corresponding to reconstructions of scalar functions and of their gradient.

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For a given PDE problem, convergent GDs are characterised by four properties, which can also serve as guidelines for the design of new schemes: *coercivity*, which corresponds to a discrete Poincaré inequality; *GD-consistency*, which expresses the ability of the scalar and gradient reconstructions to approximate functions in the space where the continuous problem is set; *limit-conformity*, linking the two reconstructions through an approximate integration by parts formula; *compactness*, corresponding to a discrete counterpart of the Rellich theorem.

In the recent monograph [28], several classical discretisation methods have been interpreted in the GDM framework. These include: arbitrary-order conforming, nonconforming, and mixed Finite Elements (FE) on standard meshes; arbitrary-order discontinuous Galerkin (DG) schemes in their SIPG form [1] (see, in particular, [35] on this point); various lowest-order Finite Volume methods on specific grids; lowest-order methods belonging to the Hybrid Mixed Mimetic family (see the unified presentation in [29] of the methods originally proposed in [8,27,34]) as well as nodal Mimetic Finite Differences (MFD) [9] on arbitrary polyhedral meshes; see also [4].

In this paper we present an important addition to the GDM framework: arbitrary-order Discontinuous Skeletal (DS) methods [18], characterised by globally coupled unknowns that are broken polynomials on the mesh skeleton. Specifically, the primary source of inspiration are the recently introduced Hybrid High-Order (HHO) methods for linear [22,20] and non-linear [16,17] diffusion problems, and the high-order non-conforming MFD (ncMFD) method of [41]; see also [2] for an interpretation in the Virtual Element framework and [3] for an introduction to the latter. We also cite here the Hybridizable Discontinuous Galerkin methods of [14], whose link with the former methods has been studied in [13]; see also [6] for a unified formulation. Like DG methods, DS methods support arbitrary approximation orders on general polytopal meshes. DS methods are, in addition, amenable to static condensation for linear(ised) problems, which can significantly reduce the number of unknowns in some configurations. They also have better data locality, which can ease parallel implementations. Moreover, lowest-order versions are often available that can be easily fitted into traditional Finite Volume simulators. Finally, unlike DG methods, DS methods admit a Fortin operator in general meshes, a crucial property in the context of incompressible or quasi-incompressible problems in solid- and fluid-mechanics; see, e.g., [20,23].

Let a polynomial degree $k \geqslant 0$ be given. The Discontinuous Skeletal Gradient Discretisations (DSGD) studied here hinge on face unknowns that ensure the global coupling and that correspond to broken polynomials of total degree up to k on the mesh skeleton, as well as locally coupled element-based unknowns that correspond to broken polynomials of total degree up to $l \in \{k-1, k, k+1\}$ on the mesh itself. The reconstruction of scalar functions is defined in a straightforward manner through the latter if $l \geqslant 0$, or by a suitable combination of face-based unknowns if $l = -1$. The gradient reconstruction, on the other hand, requires a more careful design. The seminal ideas to devise high-order gradient reconstructions on general meshes are already present, among others, in HHO methods (see, e.g., [22, Eq. (13)] and [16, Eq. (4.3)]) as well as in ncMFD methods (see [41, Eq. (21)]). These gradient reconstructions, however, are not suitable to define a convergent DSGD because they fail to satisfy the coercivity requirement. In addition, when considering non-linear problems, the codomain of the gradient reconstruction has to be carefully selected in order for the GD-consistency requirement to be satisfied with optimal scaling in the meshsize for $k \geqslant 1$ (this point was already partially recognised in [16]). In the context of DG methods, a stable discrete gradient based on a variation of the method originally proposed in [12] has been recently studied in [42].

The main novelty of this work is the introduction of a gradient reconstruction that meets all the requirements to define a convergent GD, and which satisfies the limit-conformity property with an error that scales optimally in the meshsize. This gradient reconstruction is composed of two terms: a consistent contribution closely inspired by [16, Eq. (4.3)] and a stabilisation term. Two design conditions are identified for the stabilisation term: (i) local stability and boundedness with respect to a suitable boundary seminorm and (ii) L^2-orthogonality to vector-valued polynomials of degree up to k. When considering problems posed in a non-Hilbertian setting, an additional condition is added stipulating that the stabilisation is built on a piecewise polynomial space. An example of stabilisation term that meets all of the above requirement is proposed based on a Raviart–Thomas–Nédélec space on a submesh.

The rest of the paper is organised as follows. In Section 2 we recall the basics of the GDM and give a few examples of linear and non-linear problems for which GDs are convergent under the coercivity, GD-consistency, limit-conformity, and compactness properties discussed above. The construction of arbitrary-order DSGD is presented in Section 3, the main results are stated in Section 3.5, and numerical examples are collected in Section 3.7. The links with HHO, ncMFD, and ncVEM schemes are studied in detail in Section 4. Appendix A contains the proofs of the main results. The material is organised so that multiple levels of reading are possible: readers mainly interested in the numerical recipe and results can primarily focus on Sections 2–3; readers also interested in the relations with other polytopal methods can consult Section 4.

2. The Gradient Discretisation Method

We give here a brief presentation of the Gradient Discretisation Method (GDM) in the context of homogeneous Dirichlet boundary conditions, and we refer to the monograph [28] for more details and other boundary conditions.

2.1. Gradient discretisations and gradient schemes

Let Ω be a bounded polytopal domain in $\mathbb{R}^d$, where $d \geqslant 1$ is the space dimension. We consider elliptic or parabolic problems whose weak formulation is set in $W_0^{1,p}(\Omega)$, where $p \in (1, +\infty)$ denotes a Sobolev exponent which we assume fixed in what follows.

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