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Numerical solution of the wave equation with variable wave speed on nonconforming domains by high-order difference potentials ☆

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ABSTRACT

We solve the wave equation with variable wave speed on nonconforming domains with fourth order accuracy in both space and time. This is accomplished using an implicit finite difference (FD) scheme for the wave equation and solving an elliptic (modified Helmholtz) equation at each time step with fourth order spatial accuracy by the method of difference potentials (MDP). High-order MDP utilizes compact FD schemes on regular structured grids to efficiently solve problems on nonconforming domains while maintaining the design convergence rate of the underlying FD scheme. Asymptotically, the computational complexity of high-order MDP scales the same as that for FD.

1. Introduction

We consider an initial boundary value problem for the wave (d'Alembert) equation:

$$u_{tt} = c^2 \Delta u + F, \qquad \overrightarrow{x} \in \Omega \tag{1a}$$

$$u\left(\overrightarrow{x}, 0\right) = \phi_0\left(\overrightarrow{x}\right) \tag{1b}$$

$$u_t\left(\overrightarrow{x}, 0\right) = \phi_1\left(\overrightarrow{x}\right) \tag{1c}$$

$$\ell(u)|_\Gamma = \psi(t) \tag{1d}$$

where $\Gamma = \partial\Omega$ is the boundary, the wave speed c is a variable function of the spatial coordinates (assumed smooth in the current work, although this limitation can be lifted as explained in Section 5), and F is an inhomogeneous term. The

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boundary condition (1d) in this work is taken to be either Dirichlet ($\ell = 1$) or Neumann ($\ell = \frac{\partial}{\partial \boldsymbol{n}}$). In our earlier work [1,2] that discussed the Helmholtz equation (i.e., the time-harmonic wave equation), we have considered a variety of more general boundary conditions as well.

Equation (1a) is an established model for a broad range of problems in acoustics and electromagnetism. The key feature of all these problems is their linearity. The numerical methods that we are developing hereafter are not designed for solving the nonlinear problems. We rather consider our main challenge as to compute the solution over large and generally shaped regions with high fidelity and robustness.

Finite difference (FD) methods are known to lead to inexpensive and efficient algorithms for computing smooth solutions on regular domains/grids. Their primary disadvantage is in dealing with more complicated geometries and solutions with low regularity. The finite element method (FEM) and its extensions, as well as the discontinuous Galerkin method (DG), may help alleviate these two constraints pertinent to FD. Yet in practical problems of wave propagation, especially in 3D, both FD and FEM have serious limitations because of their relatively high "points-per-wavelength" requirement, as well as numerical pollution (the dispersion error), see [3,4] and [5, Section 4.6.1]. The numerical phase velocity of the wave in these methods depends on the wavenumber. Therefore, a propagating packet of waves with different frequencies gets distorted in the simulation. Furthermore, the numerical error strongly depends on the frequency [6,5].

This drawback can be (partially) overcome by high-order FD schemes. They, however, usually need a wider stencil, which complicates the boundary conditions. A class of schemes aimed at reducing the phase error are the dispersion relation preserving schemes [7,8]. Yet they need an even wider stencil than conventional schemes of the same order of accuracy.

There is also a special type of high-order schemes that do not require a wider stencil. These schemes rely on a targeted approximation of the class of solutions rather than of a much broader class of generic sufficiently smooth functions. The equation-based compact schemes that we have developed in [9–11] for the Helmholtz equation are in this category; other similar methods include [12–15]. A recent extension of compact equation-based schemes to the time domain is given in [16]. Such schemes reduce pollution while keeping the treatment of the boundary conditions simple. However, geometry still remains a hurdle.

In FEM, on the other hand, a high-order accurate approximation can be built for arbitrary boundaries with the help of isoparametric elements [17]. These methods require a grid generation which can be nontrivial for complex geometries and interfaces. In DG, discontinuous enrichment methods, and generalized FEM, high-order accuracy also requires additional degrees of freedom. The disadvantage of these methods for the linear problems with smooth solutions is their substantial redundancy, which entails additional computational costs.

A group of methods known to provide a very considerable flexibility from the standpoint of geometry are the boundary element methods (BEM). They typically apply to steady-state or time-harmonic problems (elliptic PDEs). In these methods, linear boundary value problems are reduced to boundary integral equations (BIE) with respect to equivalent boundary sources. BEM impose practically no limitations on the shape of the boundary and automatically account for the correct far field behavior of the solution. However, these methods rely on the explicit knowledge of the fundamental solution (and so they are not easily compatible with variable propagation speed), and the treatment of the boundary conditions requires care in choosing the boundary sources so as to maintain the equivalence of the reduction and well-posedness of the resulting boundary representation. In doing so, the cases that involve resonances of the complementary domain require special attention, see, e.g., [18].

Standard BEM cannot be used directly for unsteady problems of wave propagation (hyperbolic PDEs). Their time-dependent applications are rather limited to combined problems with a clearly identifiable elliptic component, such as slow speed flows of viscous fluid [19,20] or water waves [21].

A special class of BIEs called the retarded potential boundary integral equations (RPBIE), see [22,23], provide a venue toward extending the BEM from elliptic to hyperbolic PDEs. However, the corresponding time domain numerical methods [24–27] are not nearly as popular as their frequency domain counterparts. One difficulty is that many time domain discretizations of RPBIEs appear prone to instabilities, even if the well-posedness of the RPBIE per se can be guaranteed in the first place (some aspects of stability have recently been studied in [28]). For the most part, however, the reason is that as the time elapses the boundary extends and the computation of convolutions involved in RPBIEs that typically relies on Laplace transform methods [29,30] becomes progressively more expensive. In that regard, we also mention work [31,32] that uses RPBIEs and convolution quadratures [29,30] for the development and analysis of far-field boundary conditions.

In our earlier work on the Helmholtz equation [33,1,34,35,2], we have employed the method of difference potentials (MDP) developed by Ryaben'kii [36–39]. The MDP can be viewed as a discrete analog of Calderon's potentials and Calderon's boundary equations with projections in functional analysis [40,41]. Its capacity of handling the boundaries of general shape is comparable to that of BIEs. Yet the MDP does not require fundamental solutions and automatically guarantees the equivalence of the reduced boundary problem and the original one. It uses discretizations on regular structured grids and can maintain high-order accuracy for non-conforming boundaries. Difference potentials for the Helmholtz equation [33,1,34,35,2] were built using compact equation-based schemes [9–11] that enable high-order accuracy while avoiding the extensive redundancy inherent in high-order FEM and DG methods.

In the current paper, we extend the previously developed MDP-based approach for time-harmonic waves to the genuinely time-dependent formulation (1). Our goal is to achieve the same geometric flexibility and high-oder accuracy as we have obtained for the Helmholtz equation [33,1,34]. Fundamentally, there may be two ways of pursuing this goal. One can build a full-fledged MDP algorithm in 3+1 dimensional space-time. In doing so, like in the case of RPBIEs, computing the operators