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Model reduction method using variable-separation for stochastic saddle point problems

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ABSTRACT

In this paper, we consider a variable-separation (VS) method to solve the stochastic saddle point (SSP) problems. The VS method is applied to obtain the solution in tensor product structure for stochastic partial differential equations (SPDEs) in a mixed formulation. The aim of such a technique is to construct a reduced basis approximation of the solution of the SSP problems. The VS method attempts to get a low rank separated representation of the solution for SSP in a systematic enrichment manner. No iteration is performed at each enrichment step. In order to satisfy the inf–sup condition in the mixed formulation, we enrich the separated terms for the primal system variable at each enrichment step. For the SSP problems by regularization or penalty, we propose a more efficient variable-separation (VS) method, i.e., the variable-separation by penalty method. This can avoid further enrichment of the separated terms in the original mixed formulation. The computation of the variable-separation method decomposes into offline phase and online phase. Sparse low rank tensor approximation method is used to significantly improve the online computation efficiency when the number of separated terms is large. For the applications of SSP problems, we present three numerical examples to illustrate the performance of the proposed methods.

1. Introduction

Saddle point problems often arise in a variety of applications in science and engineering. For example, mixed finite element methods in engineering application such as fluid and solid mechanics are the typical examples of saddle point systems [6,36], and quadratic programming in optimal control is another popular application [27], and so on. The measurement noise or the lack of knowledge about the physical properties usually brings uncertainties to the model inputs. The uncertainties are often parameterized by random variables. In this work, we consider the stochastic saddle point problems and discuss the related applications.

In the last two decades, spectral stochastic methods (e.g., [19,31,20]) have been extensively investigated to explore the uncertainty propagation for the complex physical and engineering systems. Most of these approaches attempt to find a functional expansion for the random solution on a suitable set of basis functions of basic random variables. To compute the approximate solution, many numerical methods, such as L^2 projection [21], Galerkin projections [2,14,30], regression [5] and stochastic interpolation [1,34,47,46,16], have been proposed. The numerical simulation is challengeable when the physical

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model has high dimensional random inputs. In order to reduce the computation complexity, model reduction methods (see [31] for a short review) have been proposed. The main idea of model reduction methods is to construct an approximate model with lower dimensionality but still describes important aspects of the original model. The reduced basis method is one of the model order reduction methods, and usually provides an efficient and reliable approximation of input–output relationship induced by parameterized partial differential equations (PDEs) [36,39,24]. These approaches have been applied to the parameterized saddle point problems [37,18,25,42,40,38,35].

Another class of model reduction methods are based on variable-separation (VS) method. As an example, proper generalized decomposition method (PGD) has been used in solving stochastic partial differential equations (SPDEs) [31–33,8,15]. The PGD method constructs optimal reduced basis from a double orthogonality criterium [33], and it requires the solutions of a few uncoupled deterministic problems solved by classical deterministic solution techniques, and the solutions of stochastic algebraic equations solved by classical spectral stochastic methods. We note that PGD requires many iterations with the arbitrary initial guess to compute each term in the separated expansion at each enrichment step. This will deteriorate the simulation efficiency. PGD has been well developed for several classes of partial differential equations, but for the parameterized saddle point problems, there are additional difficulties, which have not been fully addressed. In this paper, we propose a new variable-separation (VS) method for the parametrized/stochastic saddle point problems to get a separated representation for the solution without iterations at each enrichment step.

Let D denote a bounded physical domain. $\mathcal{V}$ and Q are two Hilbert spaces defined on D with inner products $(\cdot,\cdot)_{\mathcal{V}}$ and $(\cdot,\cdot)_Q$, respectively. Their associated norms $\|\cdot\|^2_{\mathcal{V}}=(\cdot,\cdot)_{\mathcal{V}}$, $\|\cdot\|^2_Q=(\cdot,\cdot)_Q$. In order to address the main idea of the proposed approach, we consider the following variational problem: for $\forall\boldsymbol{\xi}\in\Omega$, we find $\{u(\boldsymbol{\xi}),p(\boldsymbol{\xi})\}\in\mathcal{V}\times Q$ such that

$$\begin{cases} a\big(u(\boldsymbol{\xi}),v;\boldsymbol{\xi}\big)+b\big(v,p(\boldsymbol{\xi});\boldsymbol{\xi}\big)=f(v;\boldsymbol{\xi}) & \forall\, u\in\mathcal{V}\\ \qquad\qquad b\big(u(\boldsymbol{\xi}),q;\boldsymbol{\xi}\big)=g(q;\boldsymbol{\xi}) & \forall\, q\in Q, \end{cases} \tag{1.1}$$

where $\boldsymbol{\xi}:=(\xi_1,\cdots,\xi_d)$ is a set of d real-valued random variables, $a(\cdot,\cdot;\boldsymbol{\xi}):\mathcal{V}\times\mathcal{V}\longrightarrow R$ and $b(\cdot,\cdot;\boldsymbol{\xi}):\mathcal{V}\times Q\longrightarrow R$ are continuous bilinear forms, $f(\cdot,\boldsymbol{\xi})$ and $g(\cdot;\boldsymbol{\xi})$ are bounded linear functionals in $\mathcal{V}$ and Q, respectively. We note that if the bilinear form $a(\cdot,\cdot;\boldsymbol{\xi})$ is symmetric for any $\boldsymbol{\xi}\in\Omega$, the solution $\{u(\boldsymbol{\xi}),p(\boldsymbol{\xi})\}$ to (1.1) corresponds to a saddle point of the Lagrangian functional, i.e.,

$$\hat{\mathcal{L}}(\boldsymbol{\xi})=\mathcal{L}(u,p,\boldsymbol{\xi})=\inf_{v\in\mathcal{V}}\sup_{q\in Q}\mathcal{L}(v,q,\boldsymbol{\xi}),$$

where

$$\mathcal{L}(v,q,\boldsymbol{\xi})=\frac{1}{2}a(v,v;\boldsymbol{\xi})+b(v,q;\boldsymbol{\xi})-f(v;\boldsymbol{\xi})-g(q;\boldsymbol{\xi}).$$

If the primal system variable $u(\boldsymbol{\xi})$ is the only interest, we may use PGD to get a separated representation for $u(\boldsymbol{\xi})$. The work in [43] presents a PGD method to solve the steady incompressible Navier–Stokes equations with random Reynolds number and forcing term. In some situations, the Lagrange multiplier $p(\boldsymbol{\xi})$ associated with the constraints has a physical interpretation. Thus the computation of $u(\boldsymbol{\xi})$ and $p(\boldsymbol{\xi})$ are both of importance. In this work, we propose a new VS method for the stochastic saddle problems, by which we can get both the separated representations for the primal system variable $u(\boldsymbol{\xi})$ and the Lagrange multiplier $p(\boldsymbol{\xi})$ simultaneously. The main idea of the VS method for SSP problems is devoted to constructing the quasi-optimal separated representations

$$u(x,\boldsymbol{\xi})\approx\sum_{i=1}^{N_u}\zeta_i^u(\boldsymbol{\xi})u^i(x)\quad\text{and}\quad p(x,\boldsymbol{\xi})\approx\sum_{i=1}^{N_p}\zeta_i^p(\boldsymbol{\xi})p^i(x)$$

in a systematic enrichment manner. At each enrichment step k, we need to solve the deterministic problem, which is induced by equation (1.1) with a fixed sample $\boldsymbol{\xi}_k$. By solving a deterministic problem one time, we can get the deterministic functions $u^{2k-1}(x)$ and $p^k(x)$, and obtain $u^{2k}(x)$, which is the Riesz representation of $b\big(v,p^k(x)\big)$. After that, we need to solve a 3×3 algebraic system to get the stochastic functions $\zeta^u_{2k-1}(\boldsymbol{\xi})$, $\zeta^u_{2k}(\boldsymbol{\xi})$ and $\zeta^p_k(\boldsymbol{\xi})$. We note that the stochastic functions $\zeta^u_{2k-1}(\boldsymbol{\xi})$, $\zeta^u_{2k}(\boldsymbol{\xi})$ and $\zeta^p_k(\boldsymbol{\xi})$ obtained in this manner depend on the previous functions $\big\{\zeta^u_{2i}(\boldsymbol{\xi}),\zeta^u_{2i-1}(\boldsymbol{\xi}),\zeta^p_i(\boldsymbol{\xi})\big\}_{i=1}^{k-1}$. This can impact on the computation efficiency and may bring challenges for numerical simulation especially when the number of terms in the separated representation is large. To avoid this issue, we construct the surrogates for $\zeta^u_{2k-1}(\boldsymbol{\xi})$, $\zeta^u_{2k}(\boldsymbol{\xi})$ and $\zeta^p_k(\boldsymbol{\xi})$ using sparse low rank tensor approximation (SLRTA) method proposed in [28] such that the surrogates $\{\hat{\zeta}^u_{2k}(\boldsymbol{\xi}),\hat{\zeta}^u_{2k-1}(\boldsymbol{\xi}),\hat{\zeta}^p_k(\boldsymbol{\xi})\}_{i=1}^N$ are independent each other. Low rank approximation methods have recently been applied to approximating functions in high dimensional tensor spaces [23,13,22], and also have been used in several applications about uncertainty propagation [10,33,26]. A low rank tensor approximations proposed in [7] requires a procedure of iterations at each step of sparse rank-one approximation with an initial guess. We extend the idea of VS to SLRTA in the work [28] to avoid the iteration procedure at each step of sparse rank-one approximation.

As we know, the penalty method is usually applied to a constraint problem such as the Stokes or Navier–Stokes equations. For the stochastic saddle problems by regularization or penalty, we propose the variable-separation by penalty (VSP)