Journal of Computational Physics 354 (2018) 86–110

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Adaptive-Mesh-Refinement for hyperbolic systems of conservation laws based on *a posteriori* stabilized high order polynomial reconstructions

Matteo Semplice [a], Raphaël Loubère [b,*]

[a] *Dipartimento di Matematica "G. Peano", Università di Torino, Via C. Alberto, 10, 10123 Torino, Italy*
[b] *Institut de Mathématiques de Bordeaux (IMB), UMR 5219, Université de Bordeaux, France*

ARTICLE INFO

Article history:
Received 8 March 2017
Received in revised form 9 October 2017
Accepted 19 October 2017
Available online xxxx

Keywords:
Adaptive Mesh Refinement
A posteriori limiter
MOOD paradigm
High order of accuracy in space and time
Entropy production
Hyperbolic conservation laws
Hydrodynamics
h/p Adaptation

ABSTRACT

In this paper we propose a third order accurate finite volume scheme based on *a posteriori* limiting of polynomial reconstructions within an Adaptive-Mesh-Refinement (AMR) simulation code for hydrodynamics equations in 2D. The *a posteriori* limiting is based on the detection of problematic cells on a so-called candidate solution computed at each stage of a third order Runge–Kutta scheme. Such detection may include different properties, derived from physics, such as positivity, from numerics, such as a non-oscillatory behavior, or from computer requirements such as the absence of NaN's. Troubled cell values are discarded and re-computed starting again from the previous time-step using a more dissipative scheme but only locally, close to these cells. By locally decrementing the degree of the polynomial reconstructions from 2 to 0 we switch from a third-order to a first-order accurate but more stable scheme. The entropy indicator sensor is used to refine/coarsen the mesh. This sensor is also employed in an *a posteriori* manner because if some refinement is needed at the end of a time step, then the current time-step is recomputed with the refined mesh, but only locally, close to the new cells. We show on a large set of numerical tests that this *a posteriori* limiting procedure coupled with the entropy-based AMR technology can maintain not only optimal accuracy on smooth flows but also stability on discontinuous profiles such as shock waves, contacts, interfaces, etc. Moreover numerical evidences show that this approach is at least comparable in terms of accuracy and cost to a more classical CWENO approach within the same AMR context.

1. Introduction

The quest for developing reliable, robust, accurate and efficient simulation codes for general hyperbolic systems of conservation laws is a work in progress since the very first simulations on computers around 1942, along with the first implemented numerical methods. Efficient simulations are based on several building-blocks or paradigms.

The first and most obvious one, is the development of robust and accurate conservative numerical methods based on the seminal works of Lax and Wendroff [16,15], Godunov [31] and many others. In this work we have chosen a third-order accurate Runge–Kutta time integration scheme and third-order accurate polynomial reconstruction with local Lax–Friedrichs

* Corresponding author.
E-mail addresses: matteo.semplice@unito.it (M. Semplice), raphael.loubere@math.u-bordeaux.fr (R. Loubère).

https://doi.org/10.1016/j.jcp.2017.10.031
0021-9991/

approximate Riemann solver: this scheme ensures a nominal third-order accuracy, while employing a small stencil of neighbors. What remains to design is a valid limiting strategy to ensure robustness and an essentially non-oscillatory behavior. In fact, any linear scheme of high accuracy inexorably develops spurious oscillations which must be damped by locally reducing the accuracy of the scheme to first order in the vicinity of any discontinuity. Most of these damping techniques rely on some sort of artificial viscosity, should it be described in a von Neumann and Richtmyer fashion [22], (slope) limiters [34–36,19,14] or stabilization techniques [12]. All have in common the desire to detect *a priori* within a solution where spurious numerical oscillations are or may appear and how much numerical dissipation is sufficient to avoid their growth. All of the above-mentioned techniques, more or less, answer these two questions, sometimes independently. In this work, instead we rely on the *a posteriori* MOOD paradigm which was developed initially in [5] and further extended to different contexts, for instance in [17,38]. This *a posteriori* procedure checks at the end of the timestep for troubled cells, and further recomputes them with a more dissipative scheme. This procedure ensures that the physical admissibility of the solution is maintained, preserves an essentially-non-oscillatory behavior and even cures pathological behaviors like unrepresentable numbers such as NaN's and Inf's.

The efficient use of the degrees of freedom on a given mesh by the numerical method is not the only factor influencing the efficiency of the numerical simulation. Another important factor is the efficient use of the available computer resources, which can be achieved via parallel computing (CPU/GPU hybrid codes, OpenMP shared memory models, MPI massively parallel distributed memory) or by the redirecting the available resolution resources (i.e. the computational cells, the degrees of freedom) to important locations of the simulation. Since this technique can be later coupled with parallelization, we focus on it for the purposes of this paper. Many different solutions have been explored since the 1970s, which fall into two main classes: adaptive mesh redistribution and adaptive mesh refinement (AMR). The redistribution implies that the code runs with a fixed number of cells but they are continuously and automatically relocated following the flow (Lagrangian or Arbitrary-Lagrangian–Eulerian schemes (ALE) [11,20]) or manually (Moving Mesh techniques [32]). Some techniques allow local changes of connectivity [18,25], or, simply, emancipate from the strict notion of mesh, like in the case of particle methods such as the Smoothed Particle Hydrodynamics (SPH) method [23]. Contrarily, the AMR strategy adds new cells where appropriate and removes old ones which are no longer required. A drastic reduction of the costs without sacrificing the level of accuracy is the main reason why those techniques were developed [2,3] and are still in use today [29]. In this work we rely on AMR technology using the so-called numerical entropy production refinement criterion [26,27,29], with a subtle but important difference compared to classical AMR procedures: each timestep is performed with the best possible refined mesh driven by the entropy production between t^n and t^{n+1}. In other words, if the numerical solution associated to its AMR mesh at time t^{n+1} could be improved (*dixit* the entropy production criterion) then, the mesh is appropriately refined, and, the solution is sent back at t^n for local re-computation. Consequently a solution is always computed with an appropriately refined mesh according to the entropy production criteria.

The main difficulty is to ensure that the association of computer technologies (parallelization and/or AMR like techniques) with a specific numerical method leads to a simulation code that is

Robust: intense shocks, wave interactions should not lead to a failure of the code;
Accurate: it is desirable to avoid inappropriate and sometimes excessive numerical diffusion and dissipation especially in regions of smooth flow;
Efficient: code architecture, parallelization environment and numerical scheme should blend to allow the efficient use of available computer resources.

When dealing with non-linear systems of PDEs, guaranteeing the robustness of high order accurate numerical schemes is increasingly difficult. Our goal in this work is to show that, at least for 2D hydrodynamics equation, the coupling of *a posteriori* limiting strategy with *a posteriori* AMR technology leads to improvements in terms of resolution capability and effectiveness. Moreover we will show that this approach leads, on the one hand, to a fail-safe numerical method for extreme situations (when violation of positivity or unrepresentable numbers may occur) and, on the other hand, to a less dissipative scheme for complex flows generating small-scale structures. This work will prove that the *a posteriori* limiting and AMR technology are relatively non-invasive to an existing serial code and also they pair together efficiently, by avoiding any prediction from data at the beginning of the timestep, but rather observing an updated numerical solution and possibly recomputing the same timestep if and where troubles have been detected.

The rest of this paper is organized as follows. The second section briefly presents the system of PDEs solved, namely Euler equations in 2D on Cartesian geometry. Next in section 3 we present the third-order accurate finite volume scheme under Adaptive Mesh Refinement (AMR) framework. Space and time accurate discretizations are described supplemented by the AMR technology driven by the numerical entropy production. Then in section 4 the *a posteriori* MOOD technique is presented as a limiting strategy to stabilize the previous third-order accurate scheme independently of the AMR procedure. The specific detection criteria, decrementing technique are discussed. This overall numerical method with MOOD limiting is further validated and tested in section 5. There, we gather the numerical results for a large set of test cases involving complex smooth and non-smooth flows. These tests assess the ability of the numerical method to maintain the optimal order of accuracy on smooth flow, and, an essentially non-oscillatory behavior on discontinuous solutions. Moreover the AMR technology permits to drastically increase the resolution of the computational mesh at little cost compared to a fully refined mesh simulation. Conclusions and perspectives are drawn in section 6.