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Dual-scale Galerkin methods for Darcy flow

Guoyin Wang [a], Guglielmo Scovazzi [a,*], Léo Nouveau [a], Christopher E. Kees [b], Simone Rossi [a], Oriol Colomés [a], Alex Main [a]

[a] Department of Civil and Environmental Engineering, Duke University, Durham, NC 27708, United States
[b] Coastal and Hydraulics Laboratory, US Army Engineer Research and Development Center, 3909 Halls Ferry Road, Vicksburg, MS 39180-6199, United States

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ABSTRACT

The discontinuous Galerkin (DG) method has found widespread application in elliptic problems with rough coefficients, of which the Darcy flow equations are a prototypical example. One of the long-standing issues of DG approximations is the overall computational cost, and many different strategies have been proposed, such as the variational multiscale DG method, the hybridizable DG method, the multiscale DG method, the embedded DG method, and the Enriched Galerkin method. In this work, we propose a mixed dual-scale Galerkin method, in which the degrees-of-freedom of a less computationally expensive coarse-scale approximation are linked to the degrees-of-freedom of a base DG approximation. We show that the proposed approach has always similar or improved accuracy with respect to the base DG method, with a considerable reduction in computational cost. For the specific definition of the coarse-scale space, we consider Raviart–Thomas finite elements for the mass flux and piecewise-linear continuous finite elements for the pressure. We provide a complete analysis of stability and convergence of the proposed method, in addition to a study on its conservation and consistency properties. We also present a battery of numerical tests to verify the results of the analysis, and evaluate a number of possible variations, such as using piecewise-linear continuous finite elements for the coarse-scale mass fluxes.

1. Introduction

Discontinuous Galerkin (DG) methods are Galerkin variational methods in which the test and trial function spaces are discontinuous polynomials. They have been applied successfully to hyperbolic problems [37,18,14] and elliptic problems, with smooth [2,1] and rough [19,41,20,39,21,38] coefficients, in both primal and mixed form [32,36,6,5,13,28,30,29,31].

DG methods have reached popularity because of their advantages in the imposition of local conservation, the enforcement of general boundary conditions and the construction of data structures for parallel implementations.

However, the application of DG methods to large-scale engineering problems has often been hampered by the larger computational cost with respect to continuous Galerkin approximations, due to the relative increase of the number of degrees-of-freedom.

* Corresponding author.
E-mail address: guglielmo.scovazzi@duke.edu (G. Scovazzi).

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Several methods have been proposed to reduce the computational cost of DG methods: the variational multiscale approach, which yields a multiscale DG method with cost comparable to a continuous Galerkin method [24,3,11]; the hybridizable DG (HDG) method [15,16,34,33], which utilizes hybridization approaches originally developed in the context of mixed finite element methods and, more recently, evolved in the embedded discontinuous Galerkin (EDG) method [17]; and, finally, the enriched Galerkin (EG) method [27], which is based on a piecewise constant discontinuous enrichment of a base continuous Galerkin method.

The variational multiscale DG method [24,3,11] was developed from the idea that the discontinuous solution can be decomposed as the sum of a continuous component and a discontinuous correction. The discontinuous correction is estimated using local condensation techniques.

The hybridizable discontinuous Galerkin methods were first developed to solve second-order elliptic problems [16]. In the HDG method, additional unknowns in the form of numerical traces are introduced at the element interfaces, and conservation is ensured by means of a global flux continuity equation. Hence all the finite elements are considered as separate subdomains and only the traces are computed directly in a global solve. The DG solution can then be post-processed solving local DG problems where the traces are enforcing the boundary conditions. HDG methods have been widely tested over different problems, however, for the lowest-order shape function, they do not reduce the computational cost. To obviate this problem, embedded discontinuous Galerkin (EDG) methods were introduced by means of a space of numerical traces that are globally continuous. However, EDG methods cannot ensure optimal convergence rates.

The enriched Galerkin (EG) method was developed by Sun and Liu [40], and is based on the idea of enriching with a piecewise constant discontinuous function a continuous Galerkin approximation, and to use the interior penalty (IP) DG framework to appropriately derive a consistent variational formulation. The piecewise-constant enrichment can be thought of as a stabilizing penalty term. The EG method has been applied to second-order elliptic equations [40], Stokes problems [4] and parabolic problems [27].

We would also like to mention the work in [26,25], where a conservative post-processing strategy was pursued to recover locally conservative fluxes from continuous Galerkin discretizations.

In this article, we propose a new dual-scale DG (DSDG) method in mixed form. We distinguish between the DG solution (or DG scale) and a coarse-scale (CS) component of the solution. A transfer operator based on local elemental problems is built to link the degrees-of-freedom between the DG and CS components of the solution. Then, the DG solution is replaced by the transfer operator applied to the CS solution in every term in the base DG variational formulation, and a new method is constructed with a significant reduction of degrees-of-freedom with respect to the original DG method. This amounts to solve the original DG formulation on the image of the transfer operator, which is a proper subspace of the DG solution space. We present a full mathematical analysis of the DSDG method and we also demonstrate its local and global conservation, robustness, stability and accuracy properties.

The article is organized as follows: after recalling the governing equations in Section 2, we define the DG notations in Section 3 and introduce a base DG method in Section 4. The DSDG approach is presented in Section 5, and an analogy with multigrid is provided in Section 6. A review of classical results in DG methods and the complete analysis of the DSDG approach are presented in Section 7 and 8, respectively. Finally, results of numerical tests are discussed in Section 9.

2. Governing equations: Darcy flow

The Darcy flow equations are a homogenized macroscopic model of transport through porous media. The mathematical structure of the Darcy flow equations is a mixed form of the Laplace equation. Consider the open set Ω in $\mathbb{R}^{n_d}$ with Lipschitz boundary $\Gamma = \partial\Omega$ ($n_d = 2, 3$ indicates the number of spatial dimensions). The Darcy flow equations are given by

$$\boldsymbol{\Lambda}^{-1}\boldsymbol{\beta} + \nabla p = \tilde{\boldsymbol{g}} \quad \text{in } \Omega, \tag{1a}$$

$$\nabla \cdot \boldsymbol{\beta} = \phi \quad \text{in } \Omega, \tag{1b}$$

$$p = p_D \quad \text{on } \Gamma_D, \tag{1c}$$

$$\boldsymbol{\beta} \cdot \boldsymbol{n} = h_N \quad \text{on } \Gamma_N, \tag{1d}$$

where $\boldsymbol{\Lambda}$ is the mobility tensor/permeability tensor, symmetric positive definite, $\tilde{\boldsymbol{g}}$ is a scaled version of the gravitational body force, that is $\tilde{\boldsymbol{g}} = \rho\,\boldsymbol{g}$, where ρ is the flow density and $\boldsymbol{g}$ is the gravity acceleration. The boundary $\Gamma = \partial\Omega$ is partitioned into the subsets Γ_D and Γ_N, on which we impose Dirichlet and Neumann Boundary conditions, and $\phi \in L^2(\Omega)$ is the source or sink term of mass in the medium. Note that we have introduced the auxiliary variable $\boldsymbol{\beta}$, representing the mass flux across the porous medium.

3. General notation and definitions of discontinuous Galerkin methods

Let $\mathscr{T}_h = \bigcup K$ be the decomposition of the domain Ω into non-overlapping, closed, shape-regular elements, such that union of elements K's covers Ω exactly and does not contains any hanging nodes [23,12]. ∂K denotes the element boundary and γ_K an edge/face of this boundary in two/three dimensions, respectively. Let h denote the mesh length scale (e.g., the