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Entropy stable high order discontinuous Galerkin methods for ideal compressible MHD on structured meshes

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ABSTRACT

We present a discontinuous Galerkin (DG) scheme with suitable quadrature rules [15] for ideal compressible magnetohydrodynamic (MHD) equations on structural meshes. The semi-discrete scheme is analyzed to be entropy stable by using the symmetrizable version of the equations as introduced by Godunov [32], the entropy stable DG framework with suitable quadrature rules [15], the entropy conservative flux in [14] inside each cell and the entropy dissipative approximate Godunov type numerical flux at cell interfaces to make the scheme entropy stable. The main difficulty in the generalization of the results in [15] is the appearance of the non-conservative "source terms" added in the modified MHD model introduced by Godunov [32], which do not exist in the general hyperbolic system studied in [15]. Special care must be taken to discretize these "source terms" adequately so that the resulting DG scheme satisfies entropy stability. Total variation diminishing / bounded (TVD/TVB) limiters and bound-preserving limiters are applied to control spurious oscillations. We demonstrate the accuracy and robustness of this new scheme on standard MHD examples.

1. Introduction

In this paper, we construct an entropy stable discontinuous Galerkin (DG) scheme for the system of ideal compressible magnetohydrodynamic (MHD) equations. As most conservation laws from applications, the entropy condition is an important property for the MHD system. It is highly desirable to design high order DG schemes to satisfy entropy stability. It is well known that a conservation law system has an entropy if and only if it is symmetrizable. Godunov [32] pointed out that systems like MHD which have a divergence constraint cannot be symmetrized unless some additional source terms (which under the divergence-free condition would be zero) are added. Therefore, we will only consider the MHD system with such source terms added.

The ideal MHD equations are a system of conservation laws for the mass, momentum, energy and magnetic field. The magnetic field has to satisfy an extra constraint that its divergence is zero, which is a reflection of the principle that there are no magnetic monopoles. The exact solution of the MHD equations preserves zero divergence for the magnetic field in future time, if the initial divergence is zero. However, in numerical computation, it is not obvious that the divergence

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free condition will be satisfied. This may lead to instabilities and loss of the positivity for density and pressure in the computations. Many works have dealt with this problem which are based on one dimensional Riemann solvers for the 7×7 system of conservation laws [7,11,12,22]. These schemes add additional steps to take care of the divergence free constraint. In [11] the authors suggest to project the numerical solution to a subspace of zero divergence solutions which involves the solution of an elliptic Poisson equation. Another method is called the constrained transport (CT) by Evans and Hawley [24], which simply means a particular finite difference discretization on a staggered grid which maintains $\nabla \cdot \mathbf{B}$ in a specific discretization. Nonstaggered versions of the CT method have also been developed, see [25,34,47]. Finite volume and discontinuous Galerkin schemes to enforce the exact divergence-free condition are developed in [2,44,4]. Another way to keep the divergence exactly zero is to rewrite the MHD equations in terms of the vector potential $\mathbf{A}$ instead of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. That not only leads to the order of spatial derivatives being increased by one, but also reduces the order of accuracy by one (see [24] for a more in-depth discussion).

A different class of schemes is the 8-wave formulation of the MHD equations suggested by Powell [46], which is found to behave better in terms of stability and accuracy than the discretization of the usual conservative form. Powell noticed that the conservative MHD equations have a Jacobian with a non-physical zero eigenvalue, and the addition of suitable non-conservative source terms would recover the correct physical structure. This approach however leads to a model that is not conservative. According to the Lax–Wendroff theorem [41], however, only conservative schemes can guarantee the correct jump conditions and propagation speed in the limit, for a discontinuous solution [52].

In terms of spatial discretizations, high order schemes have been developed by different techniques, including essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes [39,2], ADER-WENO schemes [5,3], adaptive mesh refinement (AMR) schemes [1], DG schemes with locally or globally divergence-free constraints [43,4], and central DG schemes with arbitrary order exactly divergence-free constraint [44].

The Runge–Kutta discontinuous Galerkin method is a popular category of high order numerical schemes developed in [20,19,18,21]. Jiang and Shu [38] proved a discrete entropy inequality for semi-discrete DG schemes for the square entropy for scalar conservation laws, in arbitrary dimension and on arbitrary triangulations, which is extended to symmetric systems by Hou and Liu [36]. In recent years, entropy stable schemes have been extensively studied. Tadmor [51] established the framework of entropy conservative fluxes and entropy stable fluxes, and Lefloch, Mercier and Rohde [42] provided a procedure to compute high order accurate entropy conservative fluxes. Entropy stable schemes have also been constructed either through the summation-by-parts (SBP) procedure [26,13,29] or through suitable split forms [27,29,30]. Motivated by these approaches, in [15], Chen and Shu discretize the integrals in the DG method by the Gauss–Lobatto type quadratures, resulting in the nodal formulation [35,40], then replace the fluxes inside the cell by entropy conservative fluxes, and take the fluxes across cell interfaces as the usual entropy dissipative fluxes. It is shown in [15] that DG schemes constructed in this general framework satisfy the entropy condition for the given entropy.

The entropy stable schemes have the advantages that the numerical method is nearly isentropic in smooth regions and entropy is guaranteed to be increasing across discontinuities. Thus, the numerics precisely follow the physics of the second law of thermodynamics. Another advantage of entropy stable algorithms is that one can limit the amount of dissipation added to the numerical scheme to guarantee entropy stability. Entropy stable schemes are developed by several authors [8,28,9,10,53,14]. In recent years, especially, the entropy schemes based on finite volume methods are popularly developed. Winters and Gassner [53] designed an affordable analytical expression of the numerical interface flux function that discretely preserves the entropy of the system with a particular source term [37]. Derigs et al. [23] extended this method into multi-physics, multi-scale AMR simulation with high-order in space by using spatial reconstruction techniques. Chandrashekar and Klingenberg [14] constructed the semi-discrete finite volume entropy stable schemes by using the symmetrized version of the equations as introduced by Godunov.

In this paper, we develop a semi-discrete entropy stable DG scheme for the MHD equations, following the approach in [15], on one and two dimensional Cartesian meshes. There is no conceptual difficulty to extend the scheme to three dimensions and to unstructured meshes, again following the framework in [15], however it will become more technical and we leave it for future work. An important ingredient in this general framework is the entropy conservative flux, for which we use the one in [14], originally designed for finite volume schemes. The resulting semi-discrete scheme is proved to be entropy stable, and a fully discrete scheme is obtained by using a Runge–Kutta scheme for time integration. We remark that the appearance of the non-conservative "source terms" added in the modified MHD model introduced by Godunov [32] are necessary in order for the equation to be symmetrizable and to accommodate an entropy inequality on the PDE level. Therefore, in order to establish entropy stability, we develop our scheme based on this non-conservative model, which renders additional technical challenges on the suitable DG discretization for the non-conservative source terms and its compatibility with the entropy stability. It also leaves the possibility as pointed out in [52] that it may give wrong solutions for some discontinuous problems. It will be interesting if it is possible to develop entropy stable schemes using the conservative formulation, which will be left for future investigation.

The organization of the paper is as follows. In section 2, we first recall the ideal MHD equations and its symmetrization properties and Godunov's modification. Then we construct the entropy stable DG scheme for the modified MHD equations in section 3. We provide standard MHD numerical tests in section 4. In section 5 we give a few concluding remarks and perspectives for future work. Finally, in the appendix we provide the proof for the main theorem.