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A fast accurate approximation method with multigrid solver for two-dimensional fractional sub-diffusion equation ☆

Xue-lei Lin [a], Xin Lu [a], Micheal K. Ng [b], Hai-Wei Sun [a,*]

[a] Department of Mathematics, University of Macau, Macao
[b] Department of Mathematics, Hong Kong Baptist University, Hong Kong

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ABSTRACT

A fast accurate approximation method with multigrid solver is proposed to solve a two-dimensional fractional sub-diffusion equation. Using the finite difference discretization of fractional time derivative, a block lower triangular Toeplitz matrix is obtained where each main diagonal block contains a two-dimensional matrix for the Laplacian operator. Our idea is to make use of the block ϵ-circulant approximation via fast Fourier transforms, so that the resulting task is to solve a block diagonal system, where each diagonal block matrix is the sum of a complex scalar times the identity matrix and a Laplacian matrix. We show that the accuracy of the approximation scheme is of $O(\epsilon)$. Because of the special diagonal block structure, we employ the multigrid method to solve the resulting linear systems. The convergence of the multigrid method is studied. Numerical examples are presented to illustrate the accuracy of the proposed approximation scheme and the efficiency of the proposed solver.

1. Introduction

Consider an initial-boundary value problem of two-dimensional fractional sub-diffusion equation (FSDE) [5,24]:

$$ {}_0^C\mathcal{D}_t^{\alpha} u = \nabla\cdot(p(x,y)\nabla u) + f(x,y,t),\ (x,y)\in\Omega,\ 0\le t\le T, \tag{1.1}$$

$$ u(x,y,t) = \phi(x,y,t),\ (x,y)\in\partial\Omega,\ 0\le t\le T, \tag{1.2}$$

$$ u(x,y,0) = \psi(x,y),\ (x,y)\in\bar{\Omega} = \Omega\cup\partial\Omega, \tag{1.3}$$

where $\Omega = (x_L, x_R)\times(y_L, y_R)$ is a rectangular domain, $\nabla\cdot(p(x,y)\nabla u)$ is the elliptic operator, $p(x,y)$ is a smooth positive function such that $\forall(x,y)\in\Omega$, $p(x,y)\ge p_0 > 0$ with p_0 being a constant, $\partial\Omega$ is the boundary, $f(x,y,t)$ is the source term, ${}_0^C\mathcal{D}_t^{\alpha} u$ is the Caputo's derivative of order α $(0<\alpha<1)$ with respect to t defined by

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,y,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,y,s)}{\partial s}(t-s)^{-\alpha} ds, \tag{1.4}$$

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* Corresponding author.
E-mail address: HSun@umac.mo (H.-W. Sun).

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with $\Gamma(x)$ denoting the gamma function, $\phi(x, y, t)$ and $\psi(x, y)$ are the given Dirichlet boundary condition and initial condition, respectively.

The FSDE is a class of fractional differential equation, which has been widely and successfully used in modeling of description of fractional random walk, anomalous diffusive systems, unification of diffusion and wave propagation phenomenon; see [1,4,12,17–19]. Since analytical solutions to FSDEs are often unavailable, many numerical schemes are proposed for solving sub-diffusion problems (see [5–7,21,22,24,25]). The fractional differential operators are nonlocal, which leads to a character of history dependence and universal mutuality (see [24]). Hence, the computational cost is too expensive for solving the discrete problems obtained from the FSDE. This motivates us to develop fast algorithm for numerical schemes. More precisely, the associated coefficient matrix having $N \times N$ blocks is usually of block lower triangular Toeplitz (BLTT) structure with each block being of size $M^2 \times M^2$ when certain numerical scheme is applied on the FSDE (1.1)–(1.3). In general, direct solvers for the BLTT linear system are often time consuming. For example, block forward substitution method [8] as a direct solver requires at least $\mathcal{O}(N^2M^2 + NM^4)$ operations. When N or M is large, the number of operations for direct solvers becomes very large. Therefore, direct solvers may not be considered when the matrix size is large.

Block forward substitution method combined with some efficient iterative solvers for BLTT linear systems may reduce the complexity to $\mathcal{O}(N^2M^2)$ operations and only require $\mathcal{O}(NM^2)$ storage. Alternatively, Zhang and Sun in [24] proposed the alternating direction implicit schemes for solving high dimensional FSDE whose resulting BLTT linear system can be directly solved with $\mathcal{O}(N^2M^2)$ operations and $\mathcal{O}(NM^2)$ storage requirement. Nevertheless, their method is only available for $p(x, y)$ being a constant. Meanwhile, it has lower temporal accuracy compared with the non-ADI scheme. Even so, both of the above mentioned methods are still expensive, when N is large.

In order to reduce the computational cost, recently, Lu, Pang and Sun [15] proposed an approximate inversion method (AIM) for solving BLTT linear systems, which can be applied to the one dimensional FSDEs. More precisely, the corresponding BLTT matrix is firstly approximated by the block ϵ-circulant matrix [3,14], which can be block diagonalized via fast Fourier transform (FFT). Each block is a tri-diagonal matrix for the one dimensional case. Therefore, they can be inverted easily. The accuracy of the approximation scheme is shown to be of $O(\epsilon)$ under some sufficient conditions.

In this paper, we extend the AIM to solve the two-dimensional case. As in [15], the resulting discretized BLTT matrix is approximated by the block ϵ-circulant matrix via the FFT. Unlike that in [15], however, each block is no longer a tri-diagonal matrix since it is from the two dimensional problem. Indeed, to the two dimensional case, after block diagonalizing by the FFT, the resulting diagonal block matrix is the sum of a complex scalar times the identity matrix and a Laplacian matrix. Therefore, it would be extensive to invert it directly. To lower the computational workload, we propose to exploit the multigrid method (MGM) to solve the complex scalar shifted Laplacian linear systems, and establish the convergence of the corresponding multigrid solver. We also investigate the resulting BLTT matrix to satisfy the condition which can guarantee the accuracy of approximation to be of $O(\epsilon)$.

The proposed algorithm consists of two parts. The first part is for block diagonalization. The second part is for multigrid solvers. The computational cost is of $\mathcal{O}(M^2N\log N)$ operations and the storage cost is of $\mathcal{O}(NM^2)$ storage, respectively. Numerical examples are presented to illustrate the accuracy of the proposed approximation scheme and the efficiency of the proposed multigrid solver.

The rest of this paper is organized as follows. In Section 2, we propose the approximation method for solving (1.1)–(1.3) and study the accuracy of the method. In Section 3, we use the multigrid method for solving linear systems generated by the approximation method, and analyze the convergence of MGM. In Section 4, experimental results are presented to show the accuracy and efficiency of the proposed method. Finally, some concluding remarks are given in Section 5.

2. The approximation method

In this section, we propose the AIM for solving (1.1)–(1.3) and give a sufficient condition to guarantee a high accuracy of the AIM.

For a positive integer N, let $\tau = T/N$, $t_n = n\tau$ $(0 \le n \le N)$. Define the time-grid $\{t_n | 0 \le n \le N\}$ for discretization of $[0, T]$, $\{\mathbf{u}^n = u(\cdot, t_n) | 0 \le n \le N\}$. For a given function $w(t)$ defined on $t \in [0, T]$, define grid function $\{w^n = w(t_n) | 0 \le n \le N\}$. Without loss of generality, we assume the approximation to ${}_0^C\mathcal{D}_t^\alpha w|_{t=t_n}$ to be

$$ {}_0^C\mathcal{D}_t^\alpha w|_{t=t_n} \approx D_\tau^\alpha w^n := \sum_{i=1}^{n} g_{n-i}^{(\alpha)} w^i + g^{(n,\alpha)} w^0, \quad 1 \le n \le N, \tag{2.1} $$

where $g_i^{(\alpha)}$ $(i = 0, 1, ..., N)$, $g^{(i,\alpha)}$ $(i = 1, ..., N)$ are constants dependent on α, i, and N, which vary in different finite difference schemes. We denote by $\mathbf{B} \in \mathbb{R}^{M^2 \times M^2}$, the discretization of the elliptic operator $-\nabla\cdot(p(x, y)\nabla u)$. We assume that there are M^2 spatial unknowns to be determined. Applying (2.1) and $\mathbf{B}$ to the FSDE (1.1)–(1.3), we obtain a BLTT linear system as follows:

$$ \mathbf{A}\mathbf{u} = \mathbf{b}, \tag{2.2} $$

where $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2, ..., \mathbf{u}^N)$ is the unknown to be solved, $\mathbf{b}$ is a given vector containing information about f, ϕ and ψ on grid points,