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A unified framework for mesh refinement in random and physical space

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ABSTRACT

In recent work we have shown how an accurate reduced model can be utilized to perform mesh refinement in random space. That work relied on the explicit knowledge of an accurate reduced model which is used to monitor the transfer of activity from the large to the small scales of the solution. Since this is not always available, we present in the current work a framework which shares the merits and basic idea of the previous approach but *does not* require an explicit knowledge of a reduced model. Moreover, the current framework can be applied for refinement in both random and physical space. In this manuscript we focus on the application to random space mesh refinement. We study examples of increasing difficulty (from ordinary to partial differential equations) which demonstrate the efficiency and versatility of our approach. We also provide some results from the application of the new framework to physical space mesh refinement.

1. Introduction

The prediction of the evolution of most physical systems is inevitably affected by uncertainties in the initial conditions, boundary conditions and/or parameters entering the formulation of the governing equations. The two prevalent methods for quantifying the effect of this uncertainty on the evolution of the statistics are Monte Carlo (MC) method and stochastic spectral method (see, e.g., [1–12] and references therein). An MC based method does not depend on the dimensionality of the problem. However, the convergence rate of an MC based method is relatively slow, and in order to get accurate estimates of the statistical quantities a large number of full scale simulations are required which is often prohibitively expensive. On the other hand, the stochastic spectral method leads to efficient algorithms as long as the number of sources of uncertainty (uncertainty dimensionality) is relatively small. For most realistic systems the dimensionality of uncertainty is rather large. This can lead to very serious obstacles in the application of the stochastic spectral method. Development of sparse grid techniques [13,2,14,15] as well as sparse polynomial bases [8] for the Galerkin formulation has greatly alleviated this difficulty, however these techniques heavily rely on the regularity of the solution in the random space. At the same time, not all values in the range of the sources of uncertainty contribute equally to the determination of the solution statistics. This creates the hope that if one can allocate more computational resources around the most important values of the random variables, then one can still produce reasonably accurate predictions of the solution statistics. Multi-element generalized polynomial chaos method (ME-gPC) [16,17] multi-element probabilistic collocation method (ME-PCM) [18] and multi-resolution wavelet expansion [19,20] provide the framework to obtain the statistics of the system when applying dif-

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ferent computational resources to different area (element) in random space. In order to effect the concentration of resources in the more sensitive areas of the random space (for example, the discontinuities in the random space), one needs tools for the reliable location of these sensitive areas. In particular, if we consider a mesh discretization of the random space, then one needs to be able to decide when, where and in which direction to refine this mesh. [16] provides an approach to refine the random elements and allocate the computational resources adaptively as the system evolves. Adaptive sparse grid methods [21–23] also provided a dynamical way to obtain unconstructed collocation mesh to resolve the system.

In [24] we presented a novel way of performing mesh refinement in random space. The approach was based on the use of a reduced model of the full order model, when such a reduced model is available in explicit form. The main idea was that successful mesh refinement depends on the accurate monitoring of the transfer of activity (e.g. mass, energy) from the large to the small scales of the solution. At the same time, this is exactly the role of a good reduced model, to reproduce faithfully this transfer of activity. Thus, if one has access to a good reduced model, then one can use it to perform the mesh refinement task. In [24] we provided theoretical results which established the soundness of our approach as well as numerical experiments which corroborated the theory. However, the whole framework relied on the explicit knowledge of the reduced model which is not always easy to obtain.

In the current work, we present a simple reformulation of the idea in [24] which retains the merits of the previous approach while at the same time eliminating the need for the explicit knowledge of the reduced model. In particular, we offer a way to compute the rate of transfer of activity from the large to the small scales by using appropriate quantities which can be computed *without* the explicit knowledge of the reduced model (see Section 3). The main idea is that at every step and at every element where refinement may be needed, one can obtain an expansion of the solution (in the element) on some chosen basis. The coefficients of this expansion can be used to compute the rate of transfer of activity from the large to the small scales (again within the element) and thus, decide whether there exists a need for refinement. We present results for the uncertainty quantification of three different cases of increasing complexity (linear ODE with uncertain coefficient, Kraichnan–Orszag system and Kuramoto Sivashinsky equation with an uncertain bifurcation parameter). In all three cases, the mesh refinement algorithm is capable of detecting the sensitive areas of the random space with high accuracy.

The adaptive mesh refinement framework described above can also be applied to situations where refinement is required in physical space. The only difference with the random space case is that in physical space when assigning values for the modes in the newly generated elements, the coupling between adjacent elements is required. In other words, the elements are not decoupled and the continuity (or more) of the solution across the boundary of adjacent elements should be guaranteed. In the current work we present numerical results for the mesh refinement around the location of shocks for the 1D Burgers equation as well as singularity detection for a 2D reaction–diffusion equation. In addition to the numerical examples we have included in an Appendix some analysis about the convergence properties of our mesh refinement algorithm. In particular, we have shown that our mesh refinement criterion is a reliable indicator of the onset of under-resolution. While the analysis presented here is for the case of the 1D Burgers equation, the arguments go through for the 2D reaction–diffusion equation as well as other systems.

2. Representation of uncertainty

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space, where Ω is the event space and $\mathcal{P}$ is the probability measure defined on $\mathcal{A} \subset 2^{\Omega}$, the σ-algebra of subsets of Ω (these subsets are called events). Let D be a subset of $\mathbb{R}^c (c \in \{1, 2, 3\})$ with boundary ∂D. Let $\mathcal{L}$ be an operator on D, and consider the following stochastic differential equation:

$$u_t(\mathbf{x}, t; \omega) = \mathcal{L}(\mathbf{x}, t, \omega; u), \qquad \mathbf{x} \in D. \tag{1}$$

The operator $\mathcal{L}$ usually involves spatial derivatives and can be either linear or nonlinear. In addition, $\mathcal{L}$ may depend on $\omega \in \Omega$. Appropriate initial conditions and boundary conditions sometimes involving random parameters relying on ω are assumed so that the problem (1) is well-posed $\mathcal{P}$-a.e. We also assume that for $\mathcal{P}$-a.e. solution $u := u(\mathbf{x}, t; \omega)$ is a function taking values in $\mathbb{R}$.

The random dependence operator $\mathcal{L}$ must satisfy a few important properties. The most important one is the "finite dimensional noise assumption" [9,2], that is the random input can be represented with a finite-dimensional probability space. More precisely, the random input can be represented by a finite set of random variables, e.g., $\boldsymbol{\xi}(\omega) = (\xi_1(\omega), \cdots, \xi_d(\omega))$ be a d-dimensional random vector for $\omega \in \Omega$. Then by the Doob–Dynkin lemma [25], the solution $u(\mathbf{x}, t; \omega)$ can be written as $u(\mathbf{x}, t; \boldsymbol{\xi}(\omega))$.

Any second-order stochastic process can be represented as a random variable at each spatial and temporal location. Applying the Karhunen–Loève expansion [6] along with the "finite dimensional noise assumption", a second-order stochastic process can be characterized by a finite set of mutually independent random variables. Thus without loss of generality, in this work we assume $\{\xi_i(\omega)\}_{i=1}^N$ are independent random variables. Let $\Gamma := \prod_{i=1}^d \Gamma_i$, where Γ_i is the image of $\xi_i(\Omega)$, for $i = 1, \ldots, d$. Let $\rho(\boldsymbol{\xi})$ be the probability density function (p.d.f.) of $\boldsymbol{\xi}$. Then the problem (1) can be restated as following: find $u : \bar{D} \times \Gamma \to \mathbb{R}$ such that ρ-a.e. for $\boldsymbol{\xi} \in \Gamma$, the equation

$$u_t(\mathbf{x}, t; \boldsymbol{\xi}) = \mathcal{L}(\boldsymbol{x}, t, \boldsymbol{\xi}; u), \qquad \mathbf{x} \in D, \tag{2}$$

with appropriate initial and boundary conditions.